Additive graph decompositions

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Definition (Graph decomposition)

- Graph $G = (V, E)$
- A decomposition $\mathcal{D}$ of a graph G is a collection of subgraphs of G (*blocks*) whose edges partition the edge set $E(G)$.
- One says that $\mathcal{D}$ is a (G, Γ)-design if $B \simeq \Gamma$, $\forall B \in \mathcal{D}$.

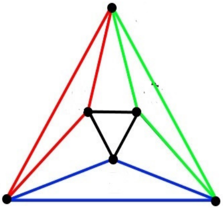

Figure: $(K_{2,2,2}, K_3)$-design.

(K_v, K_k)-design

$=$

decomposition of K_v into cliques of size k

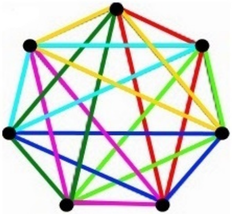

Figure: (K_7, K_3)-design.

Theorem (Kirkman, 1847)

A (K_v, K_3)-design exists if and only if $v \equiv 1, 3 \pmod 6$

Remark

$$(K_v, K_k)\text{-design}$$

$$\Leftrightarrow$$

$$2\text{-}(v, k, 1)\text{-design}$$

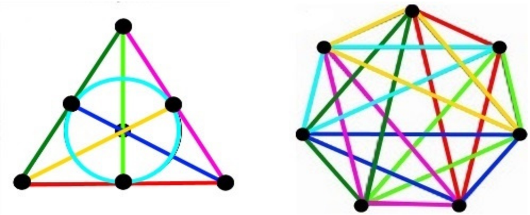

Figure: 2-(7,3,1) design and (K_7, K_3)-design.

Definition

A 2-$(v,k,1)$ *design* is a pair $(V,\mathcal{B})$ such that

- V is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of V (called blocks);
- each 2-subset of V is contained in one block.

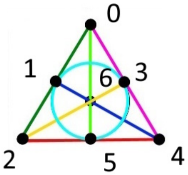

Figure: The Fano plane is 2-$(7,3,1)$ design.

- 2-$(v,3,1)$-design exists if and only if $v \equiv 1,3 \pmod 6$ (Kirkman, 1847)
- 2-$(v,4,1)$-design exists if and only if $v \equiv 1,4 \pmod{12}$ (Hanani, 1975)
- 2-$(v,5,1)$-design exists if and only if $v \equiv 1,5 \pmod{20}$ (Hanani, 1975)

- Handbook of Combinatorial Designs,
- Edited By Charles J. Colbourn, Jeffrey H. Dinitz
- Chapman & Hall, 2007, 2nd Edition

Definition (Caggegi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if

- $V \subseteq G$ and
- $\sum_{x \in B} x = 0, \quad \forall B \in \mathcal{B}.$

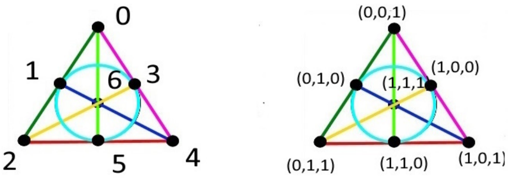

Figure: The Fano plane is additive under $\mathbb{Z}_2^3$.

- check for one block: $(0,0,1) + (0,1,0) + (0,1,1) = (0,0,0)$

Theorem (Caggegi, Falcone, Pavone, 2017)

A 2-$(v, 3, 1)$-design is additive if and only if:

- *it is the point-line design of a projective geometry PG$(n, 2)$ ($v = 2^n - 1$, additive under $\mathbb{Z}_2^n$)*

  or

- *it is the point-line design of an affine geometry AG$(n, 3)$ ($v = 3^n$, additive under $\mathbb{Z}_3^n$).*

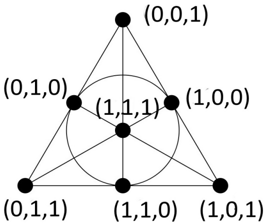

Figure: The Fano plane is PG$(2, 2)$.

[Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Description
$2\text{-}(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$	$\mathrm{AG}_1(n, p^m)$
$2\text{-}(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	$\mathrm{PG}_1(n-1, 2)$
$2\text{-}([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	$\mathrm{PG}_1(2, q)$

[Buratti, A.N., 2023, 2024]

Parameters	Group	Description
$2\text{-}(5^3, 5, 1)$	$\mathbb{F}_{5^3}$	not isomorphic to $\mathrm{AG}_1(3, 5)$
$2\text{-}(7^3, 7, 1)$	$\mathbb{F}_{7^3}$	not isomorphic to $\mathrm{AG}_1(3, 7)$
$2\text{-}(p^n, p, 1)$	$\mathbb{F}_{p^n}$	$p \in \{5, 7\}$, $n \geq 3$, not isomorphic to $\mathrm{AG}_1(n, p)$
$2\text{-}([n+1]_q, [2]_q, 1)$	$\mathbb{Z}_q^{[n+1]_q}$	$\mathrm{PG}_1(n, q)$
$2\text{-}([n+1]_q, [2]_q, 1)$	$\mathbb{F}_q^{n+1}$	$\mathrm{PG}_1(n, q)$
$2\text{-}(kq^n, k, 1)$	$G \times \mathbb{F}_q$	$k \not\equiv 2 \pmod 4$, $k \neq 2^3 \geq 12$

Definition (Buratti, Merola, A.N., 202?)

Given a simple graph Γ, a (K_v, Γ)-design *additive* under an abelian group G is a decomposition of the graph K_v into subgraphs (*blocks*) $B_1, \ldots, B_t$ all isomorphic to Γ, such that

- the vertex set $V(K_v)$ is a subset of G, and
- the sets $V(B_1), \ldots, V(B_t)$ are zero-sum in G.

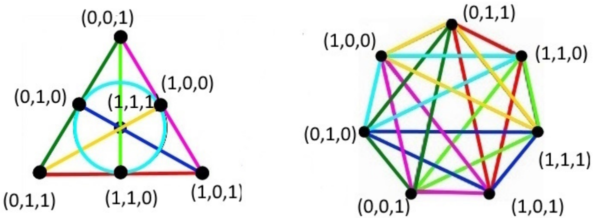

Figure: $\mathbb{Z}_2^3$-additive 2-$(7, 3, 1)$-design and (K_7, K_3)-design.

Additive Graph Decomposition

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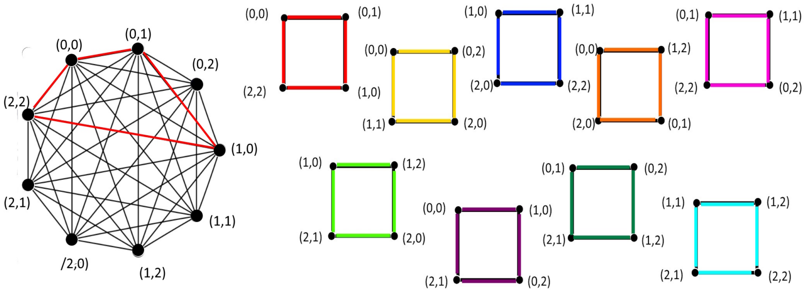


Figure: $EA(9)$-additive (K_9, C_4)-design.

Definition (Buratti, Merola, A.N., 202?)

- A subset of $\mathbb{Z}_v$ or $\mathbb{Z}_v^+ = \mathbb{Z}_v \cup \{\infty\}$ will be said *coseted* if it is partitionable into cosets of non-trivial subgroups of $\mathbb{Z}_v$, and possibly $\{\infty\}$.

Example

- $\mathbb{Z}_{12}$

$$S = \{0,1,5,6,9\} = \{0,6\} \cup \{1,5,9\} = S_1 \cup S_2$$

- $S_1 = \{0,6\}$ is the subgroup of $\mathbb{Z}_{12}$ of order 2
- $S_2 = \{1,5,9\}$ is a coset of the subgroup $\{0,4,8\}$ of $\mathbb{Z}_{12}$ of order 3

Definition (Buratti, Merola, A.N., 202?)

- A (K_v, Γ)-design is *coseted* if we have $V(K_v) = \mathbb{Z}_v$ or $\mathbb{Z}_{v-1}^+$ and the vertex set of every block is coseted.

Definition

(K_v, Γ)-design $\mathcal{D}$ is

- *cyclic* if $V(K_v) = \mathbb{Z}_v$ and every translation of $\mathbb{Z}_v$ leaves it invariant, that is to say that the group of translations of $\mathbb{Z}_v$ is an automorphism group of $\mathcal{D}$.
- *1-rotational* if $V(K_v) = \mathbb{Z}_v \cup \{\infty\}$ and the group of translations of $\mathbb{Z}_v$ is an automorphism group of $\mathcal{D}$.

Proposition

A cyclic or 1-rotational (K_v, Γ) design is coseted if and only if the vertex set of every base block is coseted.

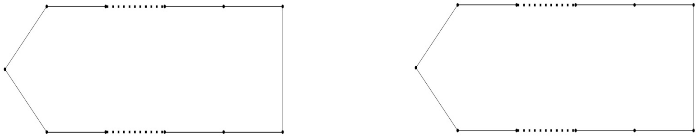
Theorem (Buratti, Merola, A.N., 202?)

Every coseted design is additive.

Theorem (Buratti, Merola, A.N., 202?)

There exists an additive (K_v, C_k)-design for any admissible pair (v, k) with $v < 3k$ and k odd, k not a prime.

- Take the 1-rotational (K_{21}, C_{15})-design $\mathcal{D}$ with base cycles $\{A, B\}$



$$A = (\infty, 0, 3, 19, 5, 18, 6, 17, 7, 16, 8, 15, 9, 13, 10);$$

$$B = (0, 1, 19, 4, 5, 3, 8, 9, 7, 12, 13, 11, 16, 17, 15).$$

- The first base cycle A is stabilized by $\{0, 10\}$
- The second base cycle B is stabilized by $\{0, 4, 8, 12, 16\}$
- So we have

$$\mathcal{D} = \{A + i \mid 0 \leq i \leq 9\} \cup \{B + i \mid 0 \leq i \leq 3\}.$$

- Let us construct an additive isomorphic copy of $\mathcal{D}$
- Consider a prime power $q \equiv 1 \pmod{20}$; we take $q = 41$.
- Take $g = 2$ as generator of the subgroup G of $\mathbb{F}_q^*$ of order 20 , that is $\mathbb{F}_{41}^{\square}$
- Consider the map $\phi^+ : \mathbb{Z}_{20}^+ \longrightarrow G^+$ defined by $\phi^+(x) = 2^x$ for every $x \in \mathbb{Z}_{20}$ and $\phi^+(\infty) = 0$.
- This map turns $\mathcal{D}$ into the isomorphic G^+-rotational (K_{21}, C_{15})-design $\phi^+(\mathcal{D})$ where

$$V(K_{21}) = G^+ = \{0, 1, 2, 4, 5, 8, 9, 10, 16, 18, 20, 21, 23, 25, 31, 32, 33, 36, 37, 39, 40\}$$

- and the base cycles are

$$A' = \phi^+(A) = (0, 1, 8, 21, 32, 31, 23, 36, 5, 18, 10, 9, 20, 33, 40),$$

$$B' = \phi^+(B) = (1, 2, 21, 16, 32, 8, 10, 20, 5, 37, 33, 39, 18, 36, 9).$$

- Thus

$$\phi^+(\mathcal{D}) = \{A' \cdot 2^i \mid 0 \le i \le 9\} \cup \{B' \cdot 2^i \mid 0 \le i \le 3\}$$

- is an isomorphic copy of $\mathcal{D}$ whose blocks are all zero-sum in $\mathbb{F}_{41}$.

[Buratti, Merola, A.N., 202?]

Parameters	Group	Description
(K_9, P_5)	$EA(9)$	
(K_9, C_4)	$EA(9)$	
(K_7, P_4)	$EA(7)$	
(K_9, P_5)	$\mathbb{Z}_{19}$	
(K_9, C_4)	$\mathbb{Z}_{19}$	
(K_{21}, M_6)	$\mathbb{Z}_{33}$	
(K_v, P_4)	$\mathbb{F}_q$, q suitable	$v \in \{7, 9, 10, 12, 13, 15, 16, 22, 24\}$
(K_{43}, P_4)	$\mathbb{Z}_{173}$	
(K_{124}, K_4)	$\mathbb{F}_{5^3}$	
(K_{30}, M_{10})	$\mathbb{Z}_{30}$	coseted
(K_{21}, C_{15})	$\mathbb{F}_{41}$	coseted
(K_{18n+10}, P_{10})	$\mathbb{Z}_{18n+10}$	$1 \leq n \leq 9$

[Buratti, Merola, A.N., 202?]

Parameters	Group	Description
(K_{9^n}, P_5)	$EA(9^n)$	
(K_{7^n}, P_4)	$EA(7^n)$	
(K_{9^n}, C_4)	$EA(9^n)$	
(K_p, C_{kp})	$EA(q)$	$q = p^n$, p prime, $q \equiv 1 \pmod{kp}$, $k \in \{2,3,4\}$
(K_p, Γ)	$EA(q)$	$q = p^n$, p prime, $q \equiv 1 \pmod{6}$, Γ generalized Petersen graph of order $2p$
(K_{mk}, C_k)	$\mathbb{Z}_k \times \mathbb{F}_m$	$k > 3$, every prime factor of m is also a factor of k
(K_v, C_k)	$\mathbb{Z}_v$	any admissible pair (v,k) with $v < 3k$ and k odd, k not a prime.
(K_{2mk}, M_{2k})	$\mathbb{Z}_{2mk}$	every pair (m,k), $k > 1$
(K_{2v}, M_{2k})	$\mathbb{Z}_{2v}$	$2v = 2mk + k + 1$ with $k \geq 3$ odd and $m > 0$

- A.N., *The first example of a simple 2-(81, 6, 2) design*. Examples and Counterexamples, 1 (2021)
- M. Buratti, A.N., *Super-regular Steiner 2-designs.* Finite Fields and Their Applications Volume 85, 102116 (2023)
- M. Buratti, A.N., *Additivity of symmetric and subspace designs,* Designs, Codes and Cryptography 92, pages 3561–3572, (2024)
- M. Buratti, F. Merola, A.N., *Additive Combinatorial Designs.* (202+)
- M. Buratti, F. Martinovic, A.N., $(27, 6, 5)$ *designs with a nice automorphism group.* (202+)

Open problem:

- applications of additive graph decompositions in other areas of mathematics and computer science?

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Thank you for your attention!