FUZZY SETS: A CONDITIONAL PROBABILITY APPROACH

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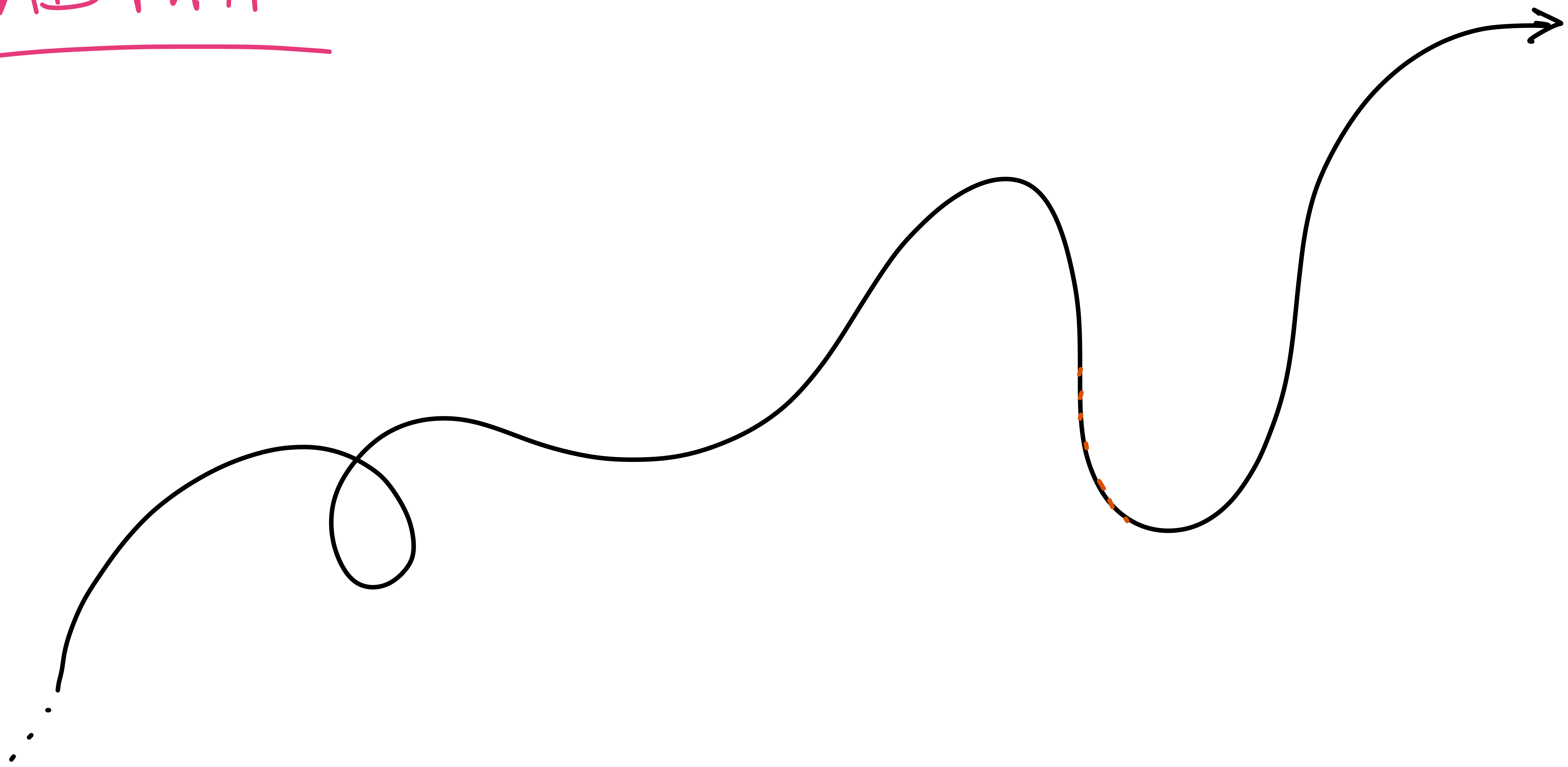
WIM2025

Palermo, 6 February 2025

ROAD MAP



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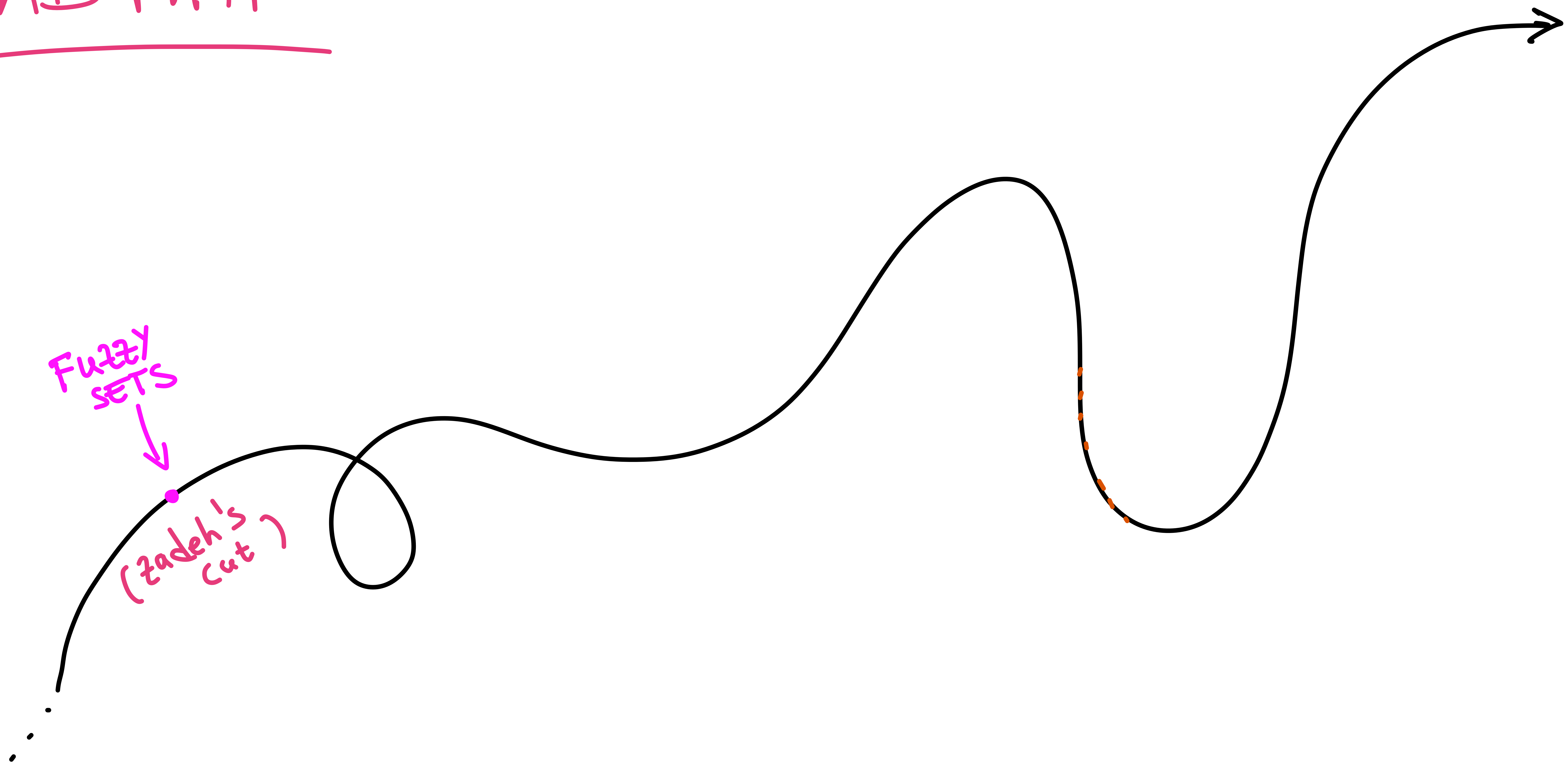

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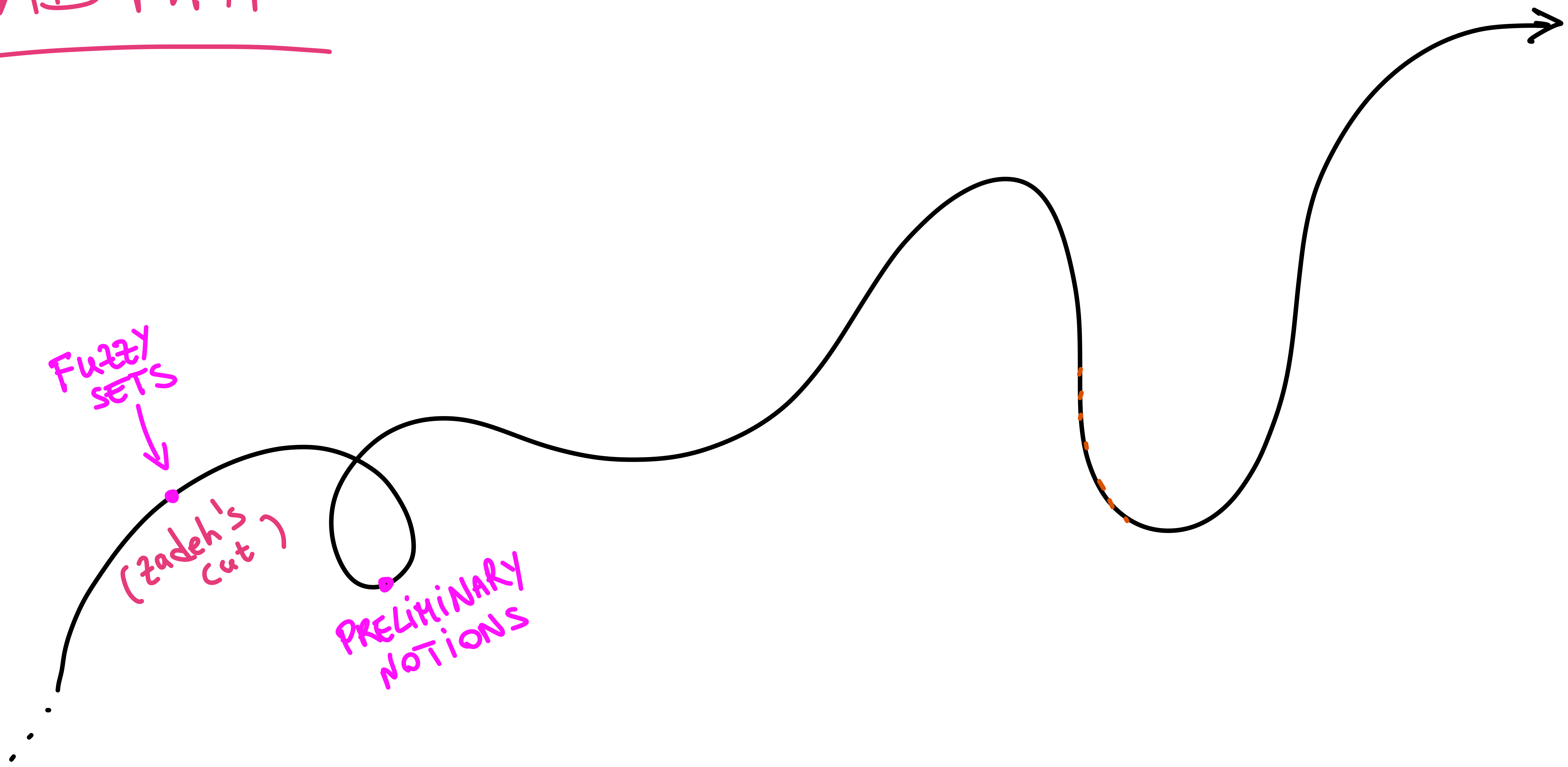

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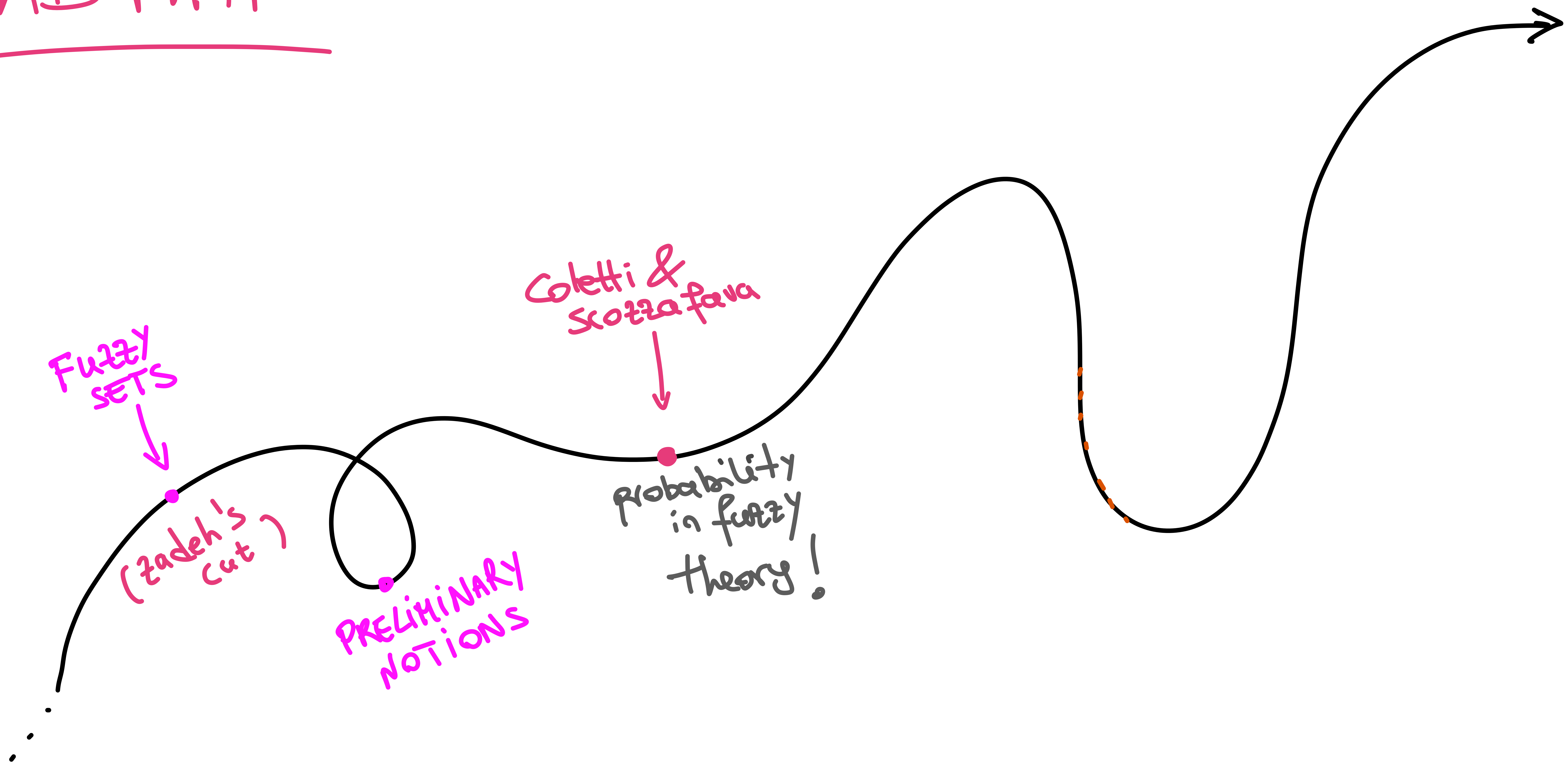
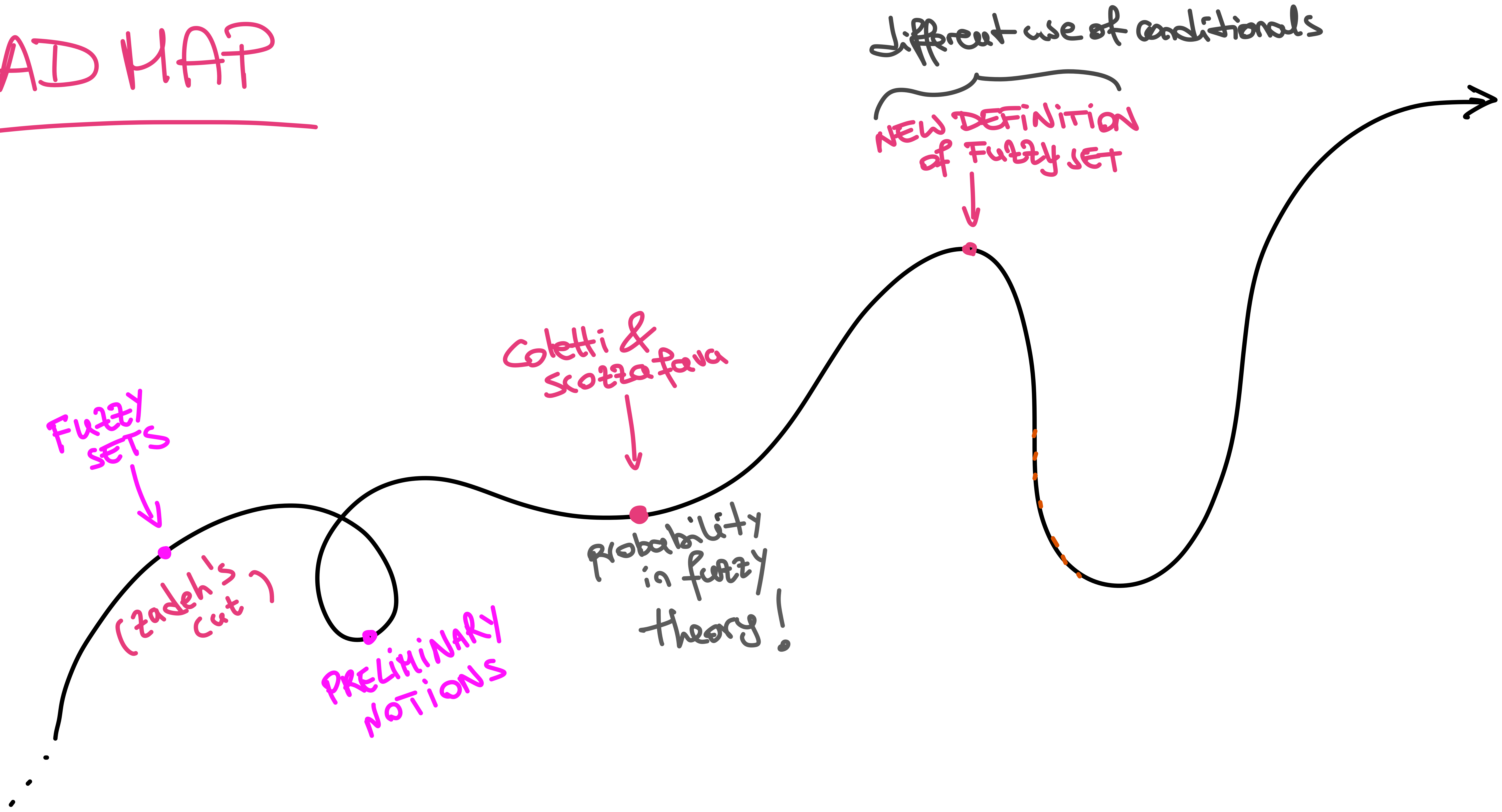


Fuzzy SETS
(Zadeh's cut)
PRELIMINARY NOTIONS
Coletti & Scozzafava
probability in fuzzy theory!

ROAD MAP



different use of conditionals
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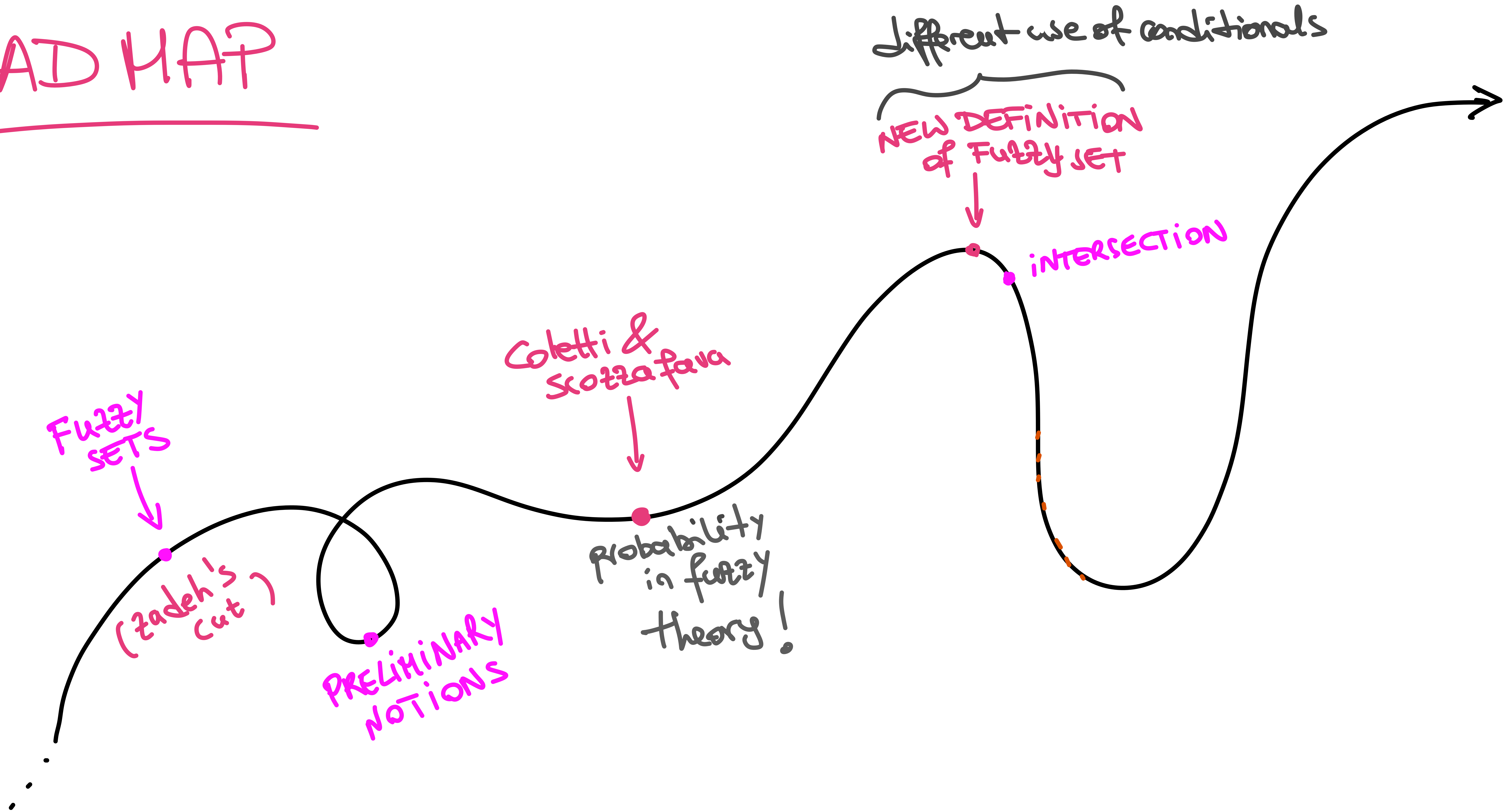
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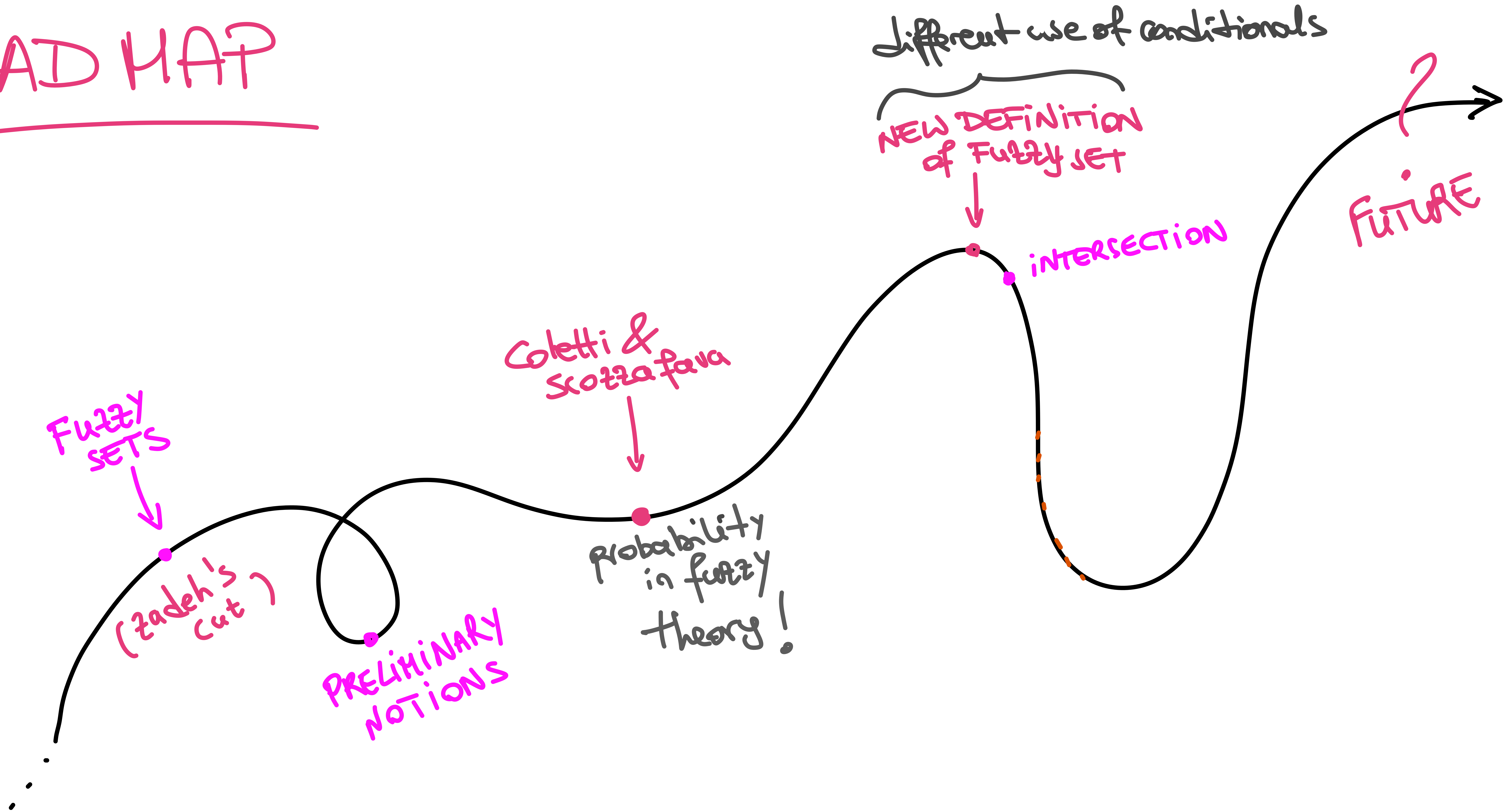

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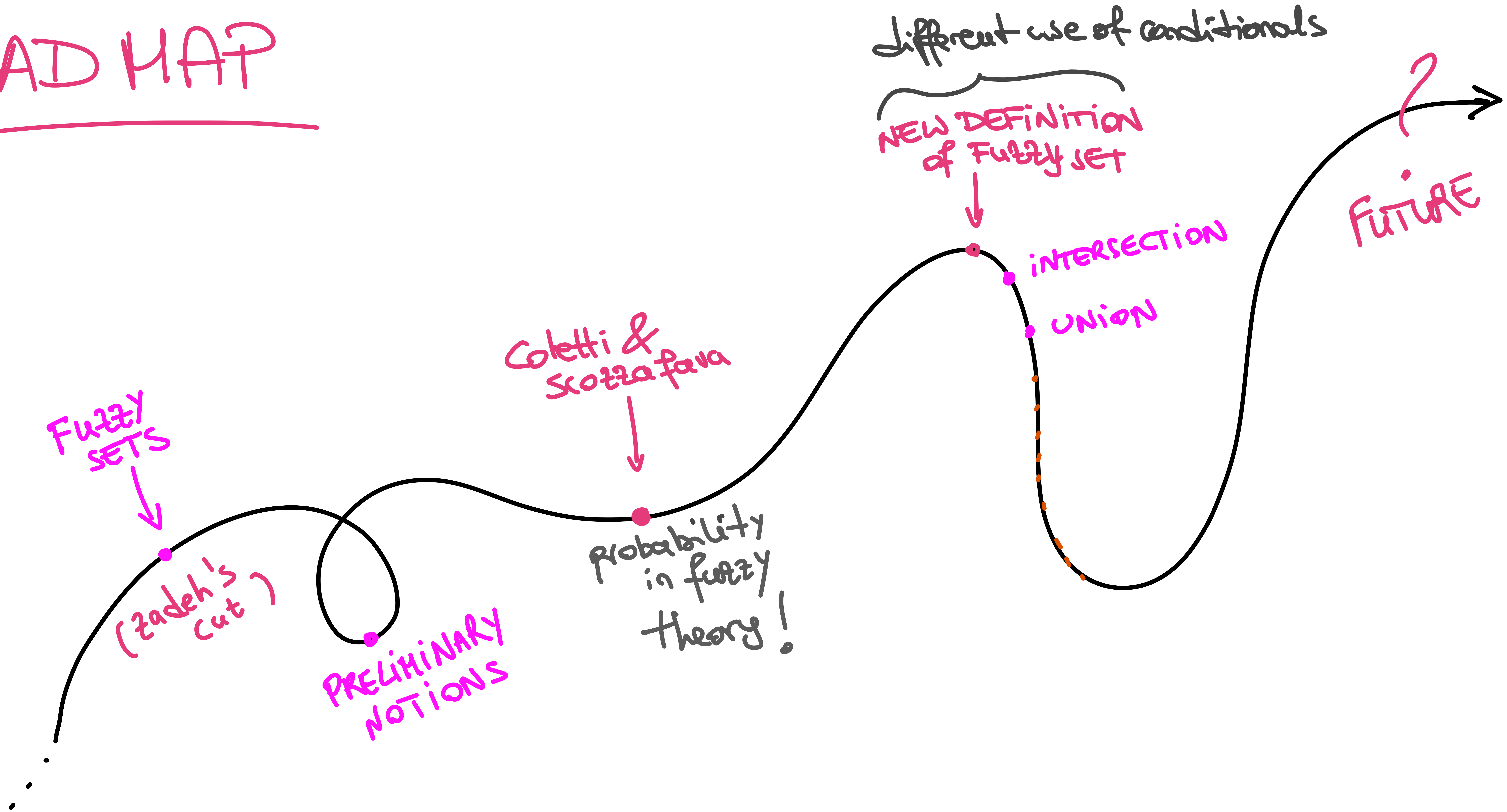

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FUZZY THEORY

Zadeh’s cut

A BRIEF HISTORY OF FUZZY THEORY

- **Fuzzy logic** is a form of many-valued logic in which the truth value of variables may be any real number between 0 and 1. It is employed to handle the concept of partial truth, where the truth value may range between completely true and completely false, in contrast with Boolean logic, the truth values of variables may only be the integer values 0 or 1.
- It is used to deal with imprecise or uncertain information and is a mathematical method for representing vagueness in decision-making.
- In Fuzzy Logic we talk of degree of truth or membership to understand "how true" a property is.
- Fuzzy logic has been applied to many fields, from control theory to artificial intelligence.

ZADEH'S DEFINITION OF FUZZY SET (1965)

Let X be a set of points, with a generic element of X denoted by x. A *fuzzy set* A in X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval [0,1]. The value of $f_A(x)$ at x representing the "grade of membership" of x in A.

$$A = \{(x, f_A(x)) \,|\, x \in X\}$$

When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A. Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set and A is called "crisp set".

FUZZY LOGIC OPERATORS

In fuzzy logic we want to work with membership values and fuzzy sets in a way that is similar to what is usually done in Boolean logic. There are several ways to replace the basic operators AND, OR, NOT. A common replacement is called the ***Zadeh operators***:

Boolean	Fuzzy
AND(x,y)	MIN(x,y)
OR(x,y)	MAX(x,y)
NOT(x)	1-x

PRELIMINARIES

Conditional events, coherence, compound conditionals…

PRELIMINARY NOTIONS

Event A: True or False values.

$$A = |A| = \begin{cases} 1 & \text{if } A \text{ TRUE} \\ 0 & \text{if } A \text{ FALSE} \end{cases}$$

Given two events A and H, we can consider their **conjunction** $A \wedge H$ and their **disjunction** $A \vee H$.

If $H \neq \emptyset$, we can define the conditional event $A \,|\, H$

The negation of a conditional event $A \,|\, H$ is $\overline{A \,|\, H} = \bar{A} \,|\, H$.

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$$A|H = \begin{cases} 1 & \text{if } AH \text{ true} \\ 0 & \text{if } \bar{A}H \text{ true} \\ x & \text{if } \bar{H} \text{ true} \end{cases} \qquad x = P(A|H)$$

The negation of a conditional event $A \,|\, H$ is $\overline{A \,|\, H} = \bar{A} \,|\, H$.

CONJUNCTION OF TWO CONDITIONAL EVENTS

Definition [Gilio, Sanfilippo 2014]

Given two conditional events $A \mid H$, $B \mid K$ and a (coherent) probability assessment $P(A \mid H) = x$, $P(B \mid K) = y$, the conjunction $(A \mid H) \wedge (B \mid K)$ is defined as the following conditional random quantity

$$(A \mid H) \wedge (B \mid K) = (AHBK + x\overline{H}BK + yAH\overline{K}) \mid (H \vee K).$$

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\,\bar{K} \text{ is true,} \end{cases}$$

$$z = \mathbb{P}((A \mid H) \wedge (B \mid K))$$

FRÉCHET-HOEFFDING BOUNDS

Theorem [Gilio, Sanfilippo, 2014]

Given any coherent assessment (x, y) on $\{A \mid H, B \mid K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset$, $K \neq \emptyset$, the extension $z = \mathbb{P}[(A \mid H) \wedge (B \mid K)]$ is coherent if and only if the Fréchet-Hoeffding bounds are satisfied, that is $z \in [z', z'']$, where

$$z' = \max\{x + y - 1, 0\},\ z'' = \min\{x, y\}.$$

Remark

In case of some logical dependencies, for the interval $[z', z'']$ of coherent extensions z it holds that $[z', z''] \subseteq [\max\{x + y - 1, 0\}, \min\{x, y\}]$.

If $HK = \emptyset \Rightarrow (A|H) \wedge (B|K) = (A|K) \cdot (B|K)$

$\Longrightarrow \quad z = z' = z'' = xy.$

COLETTI & SCOZZAFAVA‘S APPROACH

SETTING

Let X be a (not necessarily numerical) random quantity with range C_X, let A_x, for any $x \in C_X$, be the event $\{X = x\}$. The family $\{A_x\}_{x \in C_X}$ is obviously a partition of the certain event. Let φ be any property related to the random quantity X: in general it does not single out an event. Consider now the event E_φ = "You claim φ" and a coherent conditional probability $P(E_\varphi | A_x)$, looked on as a real function $\mu_{E_\varphi}(x) = P(E_\varphi | A_x)$ defined on C_X.

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MEMBERSHIP FUNCTION !

Definition [Coletti, Scozzafava, 2002,2004]

Given a random quantity X with range C_X and a related property φ, a fuzzy subset E^*_φ of C_X is the pair $E^*_\varphi = \{E_\varphi, \mu_{E_\varphi}\}$ with $\mu_{E_\varphi}(x) = P(E_\varphi | A_x)$ for every $x \in C_X$.

COLETTI & SCOZZAFAVA: PROBABILITY AND CONDITIONALS IN FUZZY THEORY

A coherent conditional probability $P(E_\varphi | A_x)$ is a measure of how much You, given the event $A_x = (X = x)$, are willing to claim the property φ, and it plays the role of the membership function of the fuzzy subset E^*_φ.

Given two fuzzy subsets E^*_φ, E^*_ψ, corresponding to the random quantities X and Y, assume that, for every $x \in C_X$ and $y \in C_Y$, the authors assume that:

- $P(E_\varphi | A_x \wedge A_y) = P(E_\varphi | A_x)$
- $P(E_\psi | A_x \wedge A_y) = P(E_\psi | A_y)$

with $A_y = \{Y = y\}$.

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Independence assumptions!

with $A_y = \{Y = y\}$.

COLETTI & SCOZZAFAVA: PROBABILITY AND CONDITIONALS IN FUZZY THEORY

"These conditions are quite natural, since they require that an event E_φ related to a fuzzy subset E^*_φ of C_X is stochastically independent (conditionally to any element of the partition $\{A_x\}_{x\in C_X}$) of every element of the partition $\{A_y\}_{y\in C_Y}$ relative to a fuzzy subset E^*_ψ of C_Y."

Definition [Coletti, Scozzafava, 2004]

Given two fuzzy subsets E^*_φ of C_X and E^*_ψ of C_Y, we define

- $E^*_\varphi \cup E^*_\psi = \{E_{\varphi\vee\psi}, \mu_{\varphi\vee\psi}\}$;
- $E^*_\varphi \cap E^*_\psi = \{E_{\varphi\wedge\psi}, \mu_{\varphi\wedge\psi}\}$;
- $(E^*_\varphi)' = \{E_{\neg\varphi}, \mu_{\neg\varphi}\}$;

where $\mu_{\varphi\vee\psi}(x,y) = P(E_\varphi \vee E_\psi | A_x \vee A_y)$ and $\mu_{\varphi\wedge\psi}(x,y) = P(E_\varphi \wedge E_\psi | A_x \wedge A_y)$ on $C_X \times C_Y$.

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where $\mu_{\varphi \vee \psi}(x, y) = P(E_\varphi \vee E_\psi | A_x \vee A_y)$ and $\overbrace{\underline{\mu_{\varphi \wedge \psi}(x, y) = P(E_\varphi \wedge E_\psi | A_x \wedge A_y)}}^{\text{Bochvar conjunction}}$ on $C_X \times C_Y$.

A SLIGHTLY MODIFIED DEFINITION

SETTING

Let X be a random quantity with range of values C_X and the event $A_x = (X = x)$, for $x \in C_X$.

Let φ be any property related to the random quantity X.

Let us consider an experiment of selecting an agent randomly from a given population.

The selected agent, for a given $x \in C_X$, must state whether the property φ holds for X or not.

E_φ = "They claim that X has the property φ".

We consider the conditional event: $E_\varphi | A_x$ = "They claim that X has the property φ, knowing that $X = x$" and a conditional probability $P(E_\varphi | A_x)$, seen as a real function $\mu_\varphi(x) = P(E_\varphi | A_x)$ defined on C_X.

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BETTING SCHEME

To look at $P(E_\varphi \mid A_x) = \mu_\varphi(x)$ in the betting framework, we consider a conditional bet that You place on the assertion of the agent randomly chosen from the given population.

That is, for every $s \in \mathbb{R}$, You pay the amount $s\mu_\varphi(x)$ in order to receive the amount:

$$\begin{cases} s, \text{ if They claim that } X \text{ has property } \varphi \text{ and } X = x; \\ 0, \text{ if They claim that } X \text{ has property } \overline{\varphi} \text{ and } X = x; \\ s\mu_\varphi(x), \text{ that is the paid amount, if } X \neq x. \end{cases}$$

FUZZY SETS

Definition

Let X be a random quantity with set of possible values C_X and a property φ on X. Given a coherent conditional probability P on the family $F_\varphi = \{E_\varphi | A_x, x \in C_X\}$, we define a fuzzy set E^*_φ in F_φ, with membership μ_φ, the set

$$E^*_\varphi = \{(E_\varphi | A_x, \mu_\varphi(x)) : x \in C_X\}, \text{ where } \mu_\varphi(x) = P(E_\varphi | A_x).$$

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- Fuzzy Emptyset $\emptyset^*$: the fuzzy set where the membership function is 0 for every $x \in C_X$.
- Fuzzy Sampleset Ω^*: the fuzzy set where the membership function is 1 for every $x \in C_X$.

COMPLEMENT OF A FUZZY SET

Negation of E_φ

$\overline{E}_\varphi$=“They claim that it is not true that X has the property φ”=“They claim that X has the property $\overline{\varphi}$”.

Thus, the negation $\overline{E}_\varphi | A_x$ of the conditional event $E_\varphi | A_x$ coincides with the conditional event $\overline{E}_\varphi | A_x$=$E_{\overline{\varphi}} | A_x$=“They claim that X has the property $\overline{\varphi}$, knowing that $X = x$”.

COMPLEMENT OF A FUZZY SET

Definition

Given a fuzzy set E^*_φ in F_φ, its complement, denoted by $\overline{E}^*_\varphi$, is the following fuzzy set in $F_{\overline{\varphi}}$:

$$\overline{E}^*_\varphi = E^*_{\overline{\varphi}} = \{(E_{\overline{\varphi}} | A_x, \mu_{\overline{\varphi}}(x)) : x \in C_X\}, \text{ where } \mu_{\overline{\varphi}}(x) = P(E_{\overline{\varphi}} | A_x).$$

Coherence requires that $\mu_{\overline{\varphi}}(x) = P(E_{\overline{\varphi}} | A_x) = P(\overline{E}_\varphi | A_x) = 1 - P(E_\varphi | A_x) = 1 - \mu_\varphi(x)$, for every $x \in C_X$. Therefore,

$$\overline{E}^*_\varphi = \{(\overline{E}_\varphi | A_x, 1 - \mu_\varphi(x)) : x \in C_X\}.$$

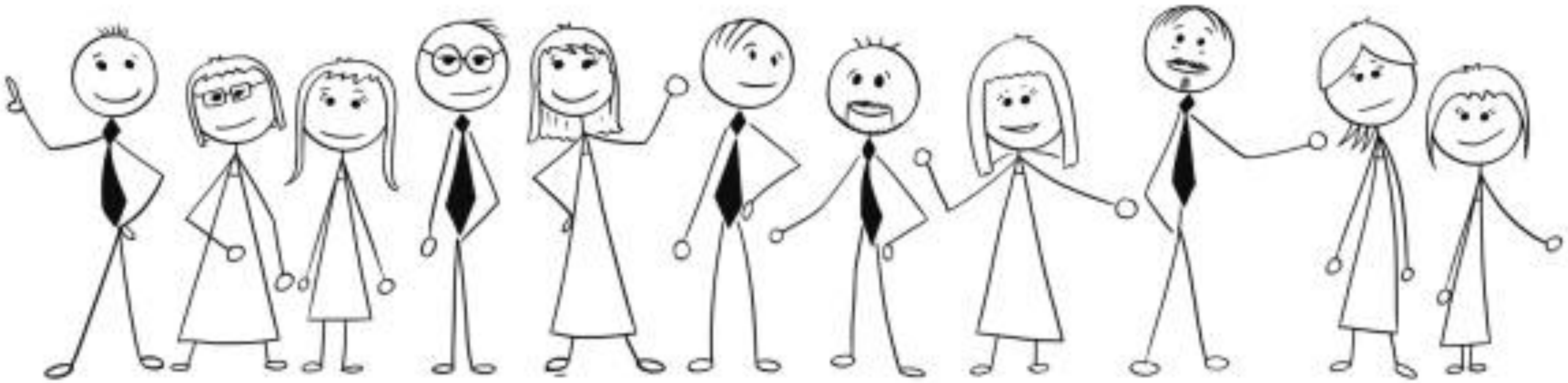
EXAMPLE

Let X denote the height (in cm) of a certain individual (let us choose Andy), with $X \in C_X = \{140,141,\ldots,220\}$. Let φ be the property "**tall**". We consider the following two events:

E_φ = "They claim that Andy is tall", A_x= "Andy's height is x cm", $x \in C_X$.

Let us consider the fuzzy set E^*_φ on C_X , where, for every $x \in C_X$, $\mu_\varphi(x) = P(E_\varphi | A_x)$ is your degree of belief on the conditional $E_\varphi | A_x$= "They claim that Andy is tall knowing that his height is x cm".

The complement $\overline{E}^*_\varphi$ of E^*_φ is the set of the pairs $(E_{\overline{\varphi}} | A_x, \mu_{\overline{\varphi}}(x)) = (\overline{E}_\varphi | A_x, 1 - \mu_\varphi(x))$, for every $x \in C_X$, where $E_{\overline{\varphi}} | A_x$ = "They claim that Andy is not tall, knowing that his height is x cm".



INTERSECTION OF FUZZY SETS

Definition

Let X, Y be two random quantities and φ, ψ two properties on X and Y, respectively. Moreover, let $E^*_\varphi = \{(E_\varphi | A_x, \mu_\varphi(x)) : x \in C_X\}$ and $E^*_\psi = \{(E_\psi | A_y, \mu_\psi(y)) : y \in C_Y\}$ be two fuzzy sets on F_φ and F_ψ, respectively. The intersection $E^*_{\varphi\wedge\psi}$ of E^*_φ and E^*_ψ is the following fuzzy set on $F_{\varphi\wedge\psi} = \{(E_\varphi | A_x) \wedge (E_\psi | A_y), (x, y) \in C_X \times C_Y\}$:

$$E^*_{\varphi\wedge\psi} = E^*_\varphi \wedge E^*_\psi = \{((E_\varphi | A_x) \wedge (E_\psi | A_y), \mu_{\varphi\wedge\psi}(x, y)) : (x, y) \in C_X \times C_Y\},$$

where $\mu_{\varphi\wedge\psi}(x, y) = \mathbb{P}[(E_\varphi | A_x) \wedge (E_\psi | A_y)]$.

$(E_\varphi | A_x) \wedge (E_\psi | A_y)$ = "They claim that: X has the property φ, knowing that $X = x$, and Y has the property ψ, knowing that $Y = y$".

DIFFERENCES WITH COLETTI & SCOZZAFAVA'S APPROACH?

C&S considered suitable assumptions of conditional independence, in particular they assumed $P(E_\varphi | (A_x \wedge A_y)) = P(E_\varphi | A_x)$ and $P(E_\psi | (A_x \wedge A_y)) = P(E_\psi | A_y)$, $(x, y) \in C_X \times C_Y$.

They defined the conjunction of $E_\varphi | A_x$ and $E_\psi | A_y$ using the Bochvar conjunction:

$(E_\varphi \wedge E_\psi) | (A_x \wedge A_y)$= **"They claim that: X has the property φ and Y has the property ψ, knowing that X = x and Y = y".**

The membership function $\mu'_{\varphi\wedge\psi}(x, y) = P((E_\varphi \wedge E_\psi) | (A_x \wedge A_y))$ $(x, y) \in C_X \times C_Y$, under conditional independence, satisfies the F-H bounds,

$$\max\{\mu_\varphi(x) + \mu_\psi(y) - 1, 0\} \leq \mu'_{\varphi\wedge\psi}(x, y) \leq \min\{\mu_\varphi(x), \mu_\psi(y)\}.$$

Indeed, without independence assumptions, for $(\mu_\varphi(x), \mu_\psi(y)) \in [0,1]^2$ on $\{E_\varphi | A_x, E_\psi | A_y\}$, every $\mu'_{\varphi\wedge\psi}(x, y) \in [0,1]$ is a coherent extension on $(E_\varphi \wedge E_\psi) | (A_x \wedge A_y)$.

IS CONDITIONAL INDEPENDENCE ALWAYS REASONABLE?

EXAMPLE

$X \in C_X = \{140,141,...,220\}$ = height (in cm) of a certain individual (let us choose Andy).

Let φ be the property "**tall**". Let us consider the fuzzy set E^*_φ on C_X and the conditional

$E_\varphi | A_x$ = "They claim that Andy is tall knowing that his height is x cm".

Let $Y \in C_Y = \{40,41,...,120\}$ denote the weight (in kg) of Andy.

Let ψ be the property "**fit**" and the events E_ψ = "They claim that Andy is fit ",
A'_y = "Andy's weight is y kg", $y \in C_Y$.

We have that every $\mu_\psi(y) \in [0,1]$ is a coherent assessment for $E_\psi | A'_y$.

IS CONDITIONAL INDEPENDENCE ALWAYS REASONABLE?

EXAMPLE

It seems natural **not to consider E_ψ stochastically independent from A_x conditionally to A'_y**, that is $P(E_\psi | A_x A'_y) \neq P(E_\psi | A'_y)$. Indeed, the information on Andy's height could modify the belief on claiming that Andy is fit only knowing his weight.

For instance, the conditional random quantity $(E_\varphi | A_{180}) \wedge (E_\psi | A'_{75})$ can be interpreted as the conjoint sentence "They claim that: Andy is tall, knowing that his height is 180 cm, and Andy is fit, knowing that his weight is 75 kg".

Suppose to assign the following values to the membership functions:

$$\mu_\varphi(180) = P(E_\varphi | A_{180}) = 0.85, \quad \mu_\psi(75) = P(E_\psi | A'_{75}) = 0.8.$$

It holds that $\mu_{\varphi\wedge\psi}(180,75) = \mathbb{P}[(E_\varphi | A_{180}) \wedge (E_\psi | A'_{75})]$ is coherent if and only if

$$\mu_{\varphi\wedge\psi}(180,75) \in [\max\{0.85 + 0.8 - 1, 0\}, \min\{0.85, 0.8\}] = [0.65, 0.8].$$

INTERESTING PROPERTIES

REMARK 1

We observe that, by the definition of intersection, it follows that $E^*_\varphi \wedge E^*_\varphi \neq E^*_\varphi$.

Let E^*_φ be a fuzzy set. Then

$$E^*_\varphi \wedge E^*_\varphi = \{((E_\varphi | A_x) \wedge (E_\varphi | A_{x'}), \mu_{\varphi\wedge\varphi}(x, x')), (x, x') \in C_X \times C_X\},$$

where $\mu_{\varphi\wedge\varphi}(x, x') = \mu_\varphi(x)$, if $x = x'$, and $\mu_{\varphi\wedge\varphi}(x, x') = \mu_\varphi(x)\mu_\varphi(x')$, if $x \neq x'$. Indeed, when $x = x'$, it holds that $(E_\varphi | A_x) \wedge (E_\varphi | A_x) = E_\varphi | A_x$ and hence by coherence $\mu_{\varphi\wedge\varphi}(x, x) = \mu_\varphi(x)$. Otherwise, when we consider pairs $(x, x') \in C_X \times C_X$ where $x \neq x'$, as $A_x A_{x'} = \emptyset$, it holds that $(E_\varphi | A_x) \wedge (E_\varphi | A_{x'}) = (E_\varphi | A_x) \cdot (E_\varphi | A_{x'})$ and hence, by coherence, $\mu_{\varphi\wedge\varphi}(x, x') = \mu_\varphi(x)\mu_\varphi(x') \leq \min\{\mu_\varphi(x), \mu_\varphi(x')\}$. Therefore, in general,

$$E^*_\varphi \wedge E^*_\varphi \neq E^*_\varphi.$$

REMARK 2

Let E_φ^* be a fuzzy set. Then, it holds that

$$E_\varphi^* \wedge \overline{E}_\varphi^* = \{((E_\varphi | A_x) \wedge (E_\varphi | A_{x'}), \mu_{\varphi\wedge\overline{\varphi}}(x, x')) : (x, x') \in C_X \times C_X\},$$

where $\mu_{\varphi\wedge\overline{\varphi}}(x, x') = 0$, if $x = x'$, and $\mu_{\varphi\wedge\overline{\varphi}}(x, x') = \mu_\varphi(x)\mu_{\overline{\varphi}}(x') = \mu_\varphi(x)(1 - \mu_\varphi(x'))$, if $x \neq x'$. Indeed, we observe that, when $x = x'$, it holds that $(E_\varphi | A_x) \wedge (E_{\overline{\varphi}} | A_x) = (E_\varphi | A_x) \wedge (\overline{E}_\varphi | A_x) = 0$ and hence by coherence $\mu_{\varphi\wedge\overline{\varphi}}(x, x) = 0$. Otherwise, when we consider pairs $(x, x') \in C_X \times C_X$ with $x \neq x'$, it holds that $(E_\varphi | A_x) \wedge (E_{\overline{\varphi}} | A_{x'}) = (E_\varphi | A_x) \cdot (E_{\overline{\varphi}} | A_{x'})$, which is not constant and can therefore not necessarily be equal to 0. Therefore, as $\mu_{\varphi\wedge\overline{\varphi}}(x, x')$ is necessarily equal to 0, it follows that

$$E_\varphi^* \wedge \overline{E}_\varphi^* \neq \emptyset^*$$

EXTENSION TO N-CONDITIONAL EVENTS

N-CONDITIONAL EVENTS

Definition

Let $X_1, \ldots, X_n$ be n random quantities and n associated properties $\varphi_1, \ldots, \varphi_n$, respectively. Let $E^*_{\varphi_i} = \{(E_{\varphi_i} | A_{x_i}, \mu_{\varphi_i}(x_i)) : x_i \in C_{X_i}\}$ be a fuzzy set on F_{φ_i}, $i = 1,\ldots,n$. The fuzzy conjunction $E^*_{\varphi_1 \wedge \ldots \wedge \varphi_n}$ is defined as

$$E^*_{\varphi_1 \wedge \ldots \wedge \varphi_n} = E^*_{\varphi_1} \wedge \ldots \wedge E^*_{\varphi_n} =$$
$$= \{((E_{\varphi_1} | A_{x_1}) \wedge \ldots \wedge (E_{\varphi_n} | A_{x_n}), \mu_{\varphi_1 \wedge \ldots \wedge \varphi_n}(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in C_{X_1} \times \ldots \times C_{X_n}\},$$

where $\mu_{\varphi_1 \wedge \ldots \wedge \varphi_n} = \mathbb{P}[(E_{\varphi_1} | A_{x_1}) \wedge \ldots \wedge (E_{\varphi_n} | A_{x_n})]$.

T-NORM

How to assign the membership function?

T-NORMS

Definition

A triangular norm (t-norm) is a binary operation T on the unit interval $[0,1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$:

(T1) $T(x, y) = T(y, x)$,

(T2) $T(x, T(y, z)) = T(T(x, y), z)$,

(T3) $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

(T4) $T(x, 1) = x$.

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$$T_M(x, y) = \min\{x, y\}$$

$$T_P(x, y) = x \cdot y$$

$$T_L(x, y) = \max\{x + y - 1,0\}$$

$$T_D(x, y) = \begin{cases} 0, \text{ if } (x, y) \in [0,1[^2 \\ \min\{x, y\}, \text{ otherwise} \end{cases}$$

FRANK T-NORMS

In our approach not all of them are acceptable to describe coherent probability for the conjunction of conditional events. A class of t-norms that in cases of logical independence describe coherent prevision assessments for the conjunction of two conditional events is given by the Frank t-norms.

For $\lambda \in [0, +\infty)$, a Frank t-norm $T_\lambda : [0,1]^2 \to [0,1]$ of parameter λ is defined as

$$T_\lambda(x, y) = \begin{cases} T_M(x, y), & \lambda = 0; \\ T_P(x, y), & \lambda = 1; \\ T_L(x, y), & \lambda = +\infty; \\ log_\lambda\left(1 + \dfrac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right), & \text{otherwise.} \end{cases}$$

They are t-norms T_λ that give a gradual transiction between the Lukasiewicz t-norm T_L and the minimum t-norm T_M, that is

$$\max\{x + y - 1, 0\} \leq T_\lambda(x, y) \leq \min\{x, y\},$$

$\lambda \in [0, +\infty]$.

FRANK T-NORM AS MEMBERSHIP FUNCTIONS

We have that, under logical independence, for every (coherent) pair of conditional probabilities $\mu_\varphi(x) = P(E_\varphi | A_x)$ and $\mu_\psi(y) = P(E_\psi | A_y)$ the assessment $\mu_{\varphi\wedge\psi}(x, y) = T_\lambda(\mu_\varphi(x), \mu_\psi(y))$ is a coherent extension on $(E_\varphi | A_x) \wedge (E_\psi | A_y)$, for every $\lambda \in [0, +\infty]$.

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$\Longrightarrow$ Frank t-norms can be used to construct the membership function of the intersection of two fuzzy sets.

⚠ The construction of membership functions by means of Frank t-norms for all the intersections which involve three fuzzy sets could lead to case of incoherent conditional prevision.

EXAMPLE 2

Given any logically independent events $E_1, E_2, E_3, H_1, H_2, H_3$, we observe that the assessment $(x_1, x_2, x_3) = (0.5, 0.6, 0.7)$ on $\{E_1 | H_1, E_2 | H_2, E_3 | H_3\}$ is coherent. Moreover, the extension

$$(T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3)) = (0.1, 0.2, 0.3, 0)$$

on the family $\{\mathscr{C}_{12}, \mathscr{C}_{13}, \mathscr{C}_{23}, \mathscr{C}_{123}\}$, where $\mathscr{C}_{ij} = (E_i | H_i) \wedge (E_j | H_j)$, and $\mathscr{C}_{123} = (E_1 | H_1) \wedge (E_2 | H_2) \wedge (E_3 | H_3)$, is not coherent.

This counterexample shows that, when considering the fuzzy sets $E^*_{\varphi_1}, E^*_{\varphi_2}, E^*_{\varphi_3}, E^*_{\varphi_1 \wedge \varphi_2}, E^*_{\varphi_1 \wedge \varphi_3}, E^*_{\varphi_2 \wedge \varphi_3}$, and $E^*_{\varphi_1 \wedge \varphi_2 \wedge \varphi_3}$, the t-norm of Lukasiewics cannot be always chosen as membership function because it could lead to incoherent assessments.

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THANKS FOR YOUR ATTENTION!!