Robustness of scale free behavior in generalized preferential attachment networks

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Complex systems and networks

Many systems involve an incredible number of components interacting in a network:

- Interactions between genes, proteins, and metabolites integrates in live cells and are a prerequisite of life.
- Neurons and their connections in the brain/nervous system determine the neural network (10^{11} nodes)
- The interactions between phones/computers (through wired internet connections or wireless links) determine communication networks
- Trades determine commercial networks
- World Wide Web (10^{12} nodes), social networks ...

$$\Downarrow$$

COMPLEX SYSTEMS

Common features of very different networks

21st century:

It becomes possible to collect data and draw maps of the evolution of networks characterized by:

- Differences: size, nature of components and of interactions, scope
- Analogies: organizing principles that allow the emerging of general laws for classes of networks

Guiding principle of studies: to uncover general organizing principles of the networks and each time verify how widely they apply.

Typical mathematical tools from the theory of Random Graphs (since 1959 paper by Paul Erdős and Alfréd Rényi).

Random network.

To model the complexity of a real system we decide where to place the links between the nodes
A random network assume that the links appear randomly between the nodes, typically:

- $G(N, p)$: fixes the probability p that two nodes are connected
- $G(N, L)$: fixes the total number of links L

A random network consists of N nodes where each node pair is connected with probability p

Click here to see image 3.3, chapter 3.

Graph Terminology

- Degree of a vertex: Number of links it has to other nodes
- Adjacency Matrix A: for a directed network of N nodes has N rows and N columns, $A_{ij} = 1$ if there is a link pointing from node i to node j and $A_{ij} = 0$, otherwise.
- Bipartite graph: is a network whose nodes can be divided into two disjoint sets X and Y such that each link connects a X-node to a Y-node.
- Path length:represents the number of links the path contains
- Clustering coefficient: accounts for the degree to which the neighbors of a given node link to each other.
- Degree Distribution: it is the probability p_k that a randomly chosen node has degree k.

Do random network fit the majority of real networks?

- In a random network implementation, nodes may exhibit very different numbers of links: this fact is highlighted by the distribution p_k.
- The degree distribution is easily computed as a $Bi(N-1,k)$ that for large N (and small p) is well approximated by a Poisson of parameter $\langle k \rangle$
- Poisson distribution implies that a large part of the nodes has degree in $\langle k \rangle \pm \sqrt{\langle k \rangle}$

This last feature conflicts with features of many real networks. Also other features of random networks show conflicts between observed data and mathematical forecast. For example, Erdős and Rényi proved that in a random network a giant component appears if and only if each node has on average more than one link but **this feature is not observed in many real networks.** Barabási A.-L. (2002). Network Science; https://networksciencebook.com/

Scale free versus random networks

A scale-free network is a network whose degree distribution follows a power law

$$p_k \sim k^{-\gamma} \tag{1}$$

Click here to see image 4.4, chapter 4.

Scale Free Networks

Many real world networks exhibit **a scale free behavior**.

- Large scale hypertexts (like WWW)
- Social networks (like Facebook)
- Financial networks (as interbank payment networks)
- Protein-protein interaction networks
- Paper citation networks
- Airline networks

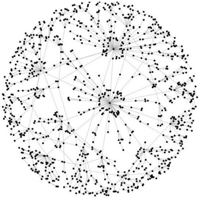
Remark

Albert and Barabási suggest an increase of robustness to perturbations of small-world networks than other network architectures. In a small-world network with a degree distribution following a power-law, it is improbable that the deletion of a random node causes a dramatic increase in mean-shortest path length.*

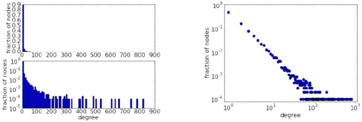
Albert R.; Barabási A.-L. (2002). "Statistical mechanics of complex networks".

Emergence of the scale free behaviour

- Why are scale free networks often observed?
- Which attachment rules determine scale free distributions?
- How robust is the scale free behaviour: does it dominate other types of asymptotics?
- What happens when we merge different attachment rules, do homogenization phenomena arise?



A scale free network



Degree distribution of a scale-free network of N=10,000 nodes and power-law exponent γ=2. First bar plot: nodes of degree larger than 100 can not be seen. Plotting the bar heights with a logarithmic scale (second bar plot) reveals the long tail of the degree distribution. Although most nodes have a very small degree, there are a few nodes with a degree above 500. These presence of hubs that are orders of magnitude larger in degree than most nodes is a characteristic of power law networks.

Preferential attachment and scale free behavior

- Barabasi and Albert* first proposed the **Preferential Attachment rule** as a cause of **Scale Free Phenomenon** in the WWW but the basic idea was already present in the classical papers by U. Yule (1925) and H.A. Simon (1955) models for different phenomena.
- Bollobás B. et al.** prove rigorously the emergence of the **Scale Free Phenomenon** in presence of **Preferential Attachment rule**

*Barabási, A.L.and Albert, R. (1999). "Emergence of scaling in random networks". Science. 286 (5439): 509–51

**Bollobás B., Riordan O., Spencer J. and Tusnády G. (2001) The degree sequence of a scalefree random graph process, Random Struct. Algorithms 18 (3) 279–290.

Robustness of the scale free property

In different papers we considered variants of 4 scale free models
Four Scale free Models

- Yule Model
- Simon Model
- Barabasi-Albert Model
- Parid Model (for suitable choice of the parameters)

We studied their robustness with respect to the scale free property in presence of:

- Detachment
- Nonlinearity/ Fractionality
- Mixed Model: Uniform and Preferential attachment
- Mixed Model: Preferential and Anti-Preferential attachment
- Parid Model (Random number of edges)

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Yule Model (Udny Yule, 1925) as a model of WWW

Reminder

A Yule process of paramenter δ is a linear birth process characterized by birth rates $\delta_n = n\delta$.

The Yule model is a continuous time process, defined through two Yule processes of parameters β and λ, resepctively. It was proposed to model the number of species per genus

- A first Yule process $\{N_\beta(T)\}_{T\geq 0}$, $\beta > 0$, accounts for the growth of the number of pages (vertices).
- As soon as the first vertex is created, a second Yule process, $\{N_\lambda(T)\}_{T\geq 0}$, $\lambda > 0$, starts describing the creation of in-links to the vertex.
- The evolution of the number of in-links for the successively created vertices, proceeds similarly.

Scale free behaviour of the Yule Model

Yule–Simon distribution

Let $\mathcal{N}_T$ be the degree of a page (vertex) chosen uniformly at random at time T. Then, if $\rho = \lambda/\beta$,

$$\lim_{T \longrightarrow \infty} \mathbb{P}(\mathcal{N}_T = k) = \rho \frac{\Gamma(k)\Gamma(1+\rho)}{\Gamma(k+1+\rho)} = \rho B(k, 1+\rho), \qquad k \geq 1.$$

The right tail of the above pmf decays as a power-law. For large k,

$$\rho B(k, 1+\rho) \approx k^{-(\rho+1)}$$

Simon model (Herbert A. Simon, 1955) as a model of WWW

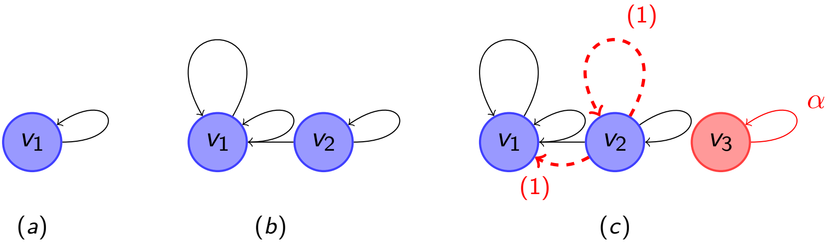


Figure: Construction of the random graph G_α^t associated to Simon model. (a) Begin at time 1 with one single vertex and a directed loop. (b) Suppose some time has passed, in this case, the picture corresponds to a realization of the process at time $t = 4$. (c) Given G_α^4 form G_α^5 by either adding with probability α a new vertex v_3 with a directed loop, or adding a directed edge with probability given by

$$\mathbb{P}(v \longrightarrow v_j) = \frac{(1-\alpha)\vec{d}(v_j, t)}{t}, \qquad 1 \leq j \leq t.$$

Main result on Simon model

Let $\vec{N}_{k,t}$ be the number of vertices with **in-degree** k at time $t = n(m+1)$, $n \in \mathbb{N}$, in the Simon model.

$$\frac{\vec{N}_{k,t}^{Simon}}{V_t} \quad \xrightarrow{\mathbb{P}} \quad \frac{1}{1-\alpha} \frac{\Gamma(k)\Gamma\left(1+\frac{1}{1-\alpha}\right)}{\Gamma\left(k+1+\frac{1}{1-\alpha}\right)} \sim \frac{1}{1-\alpha} k^{-1-\frac{1}{1-\alpha}},$$

$\Rightarrow$ It seems reasonable to look for a connection between Yule and Simon model!

Barábasi-Albert Model

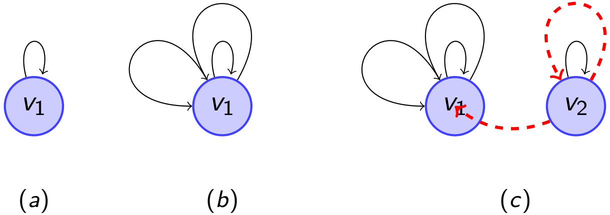


Figure: Construction of $(G_m^t)_{t\geq 1}$ for $m = 2$. (a) Begin at time 1 with one single vertex and a directed loop. (b) Suppose some time has passed, in this case, the picture corresponds to a realization of the process at time $t = 2$. Keep in mind that here $m = 2$ and therefore $m = 2$ directed edge are added to the graph by preferential attachment rule (but at this point the only possible choice is the vertex v_1). (c) Here time is $t = 4$. A new vertex v_2 already appeared at time $t = 3$ together with a directed loop. At time 4 instead the first of the m edges that must be added to the graph is chosen (red dashed directed edges) by means of the preferential attachment probabilities .

Barábasi-Albert Model

Barabási and Albert model in terms of a random graph process:

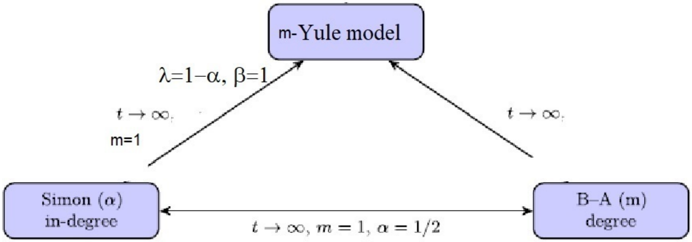
- Add at each time step a vertex with m, $m \in^{+}$, directed edges.
- If $m = 1$, $(G_1^t)_{t \geq 1}$ is a random graph process with G_1^t directed graph starting at time $t = 1$ with one vertex v_1 and one loop.
- Given G_1^t form G_1^{t+1} add the vertex v_{t+1} together with a single edge directed from v_{t+1} to v_j, $1 \leq j \leq t+1$, with probability

$$\mathbb{P}(v_{t+1} \longrightarrow v_j) = \begin{cases} \frac{d(v_j,t)}{2t+1}, & 1 \leq j \leq t, \\ \frac{1}{2t+1}, & j = t+1. \end{cases} \tag{2}$$

- For $m > 1$ define the process $(G_m^t)_{t \geq 1}$ by running the process (G_1^t) on the sequence of imaginary vertices $v_1', v_2', \ldots$, then form the graph G_m^t from G_1^{mt} by identifying the vertices $v_1', v_2', \ldots, v_m'$ to form v_1, $v_{m+1}', v_{m+2}' \ldots, v_{2m}'$ to form v_2 ...

* Bollobás, B.; Riordan, O.; Spencer, J.; Tusnády, G. (2001). "The degree sequence of a scale-free random graph process". Random Structures and Algorithms. 18 (3): 279–290.

Relations: Yule-Simon-Barabasi Albert models*



The degree of a vertex chosen uniformly at random in a B-A(m) model converges in distribution to the size of a uniformly chosen page in a *m*-Yule model (a Yule model in which m new links appear simoultaneously).

* Pachon A. Polito F. and LS (2016) Random Graphs Associated to some Discrete and Continuous Time Preferential Attachment Models. J. Stat. Phys.162, 1608-1638

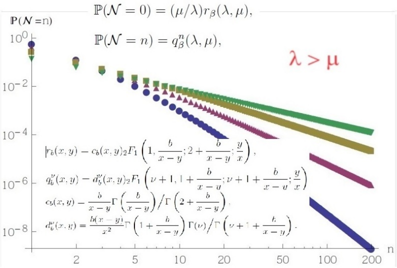
Pachon A. Polito F. and LS (2020) On the continuous-time limit of the Barabási-Albert random

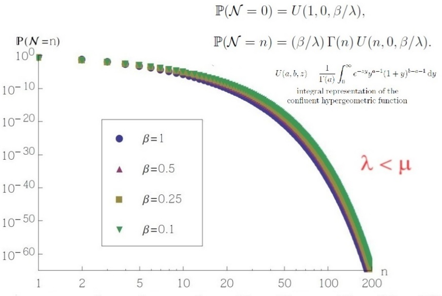
Robustness with respect to Detachment (Death)

- We study the consequences of **detachment** of in-links (**death**).
- This is accomplished by considering the Yule model and replacing the linear birth process governing the growth of in-links with a linear birth-death process.
- It turns out that the introduction of the possibility of detachment of in-links in the Yule model still leads to an analytically tractable model and thus still permits to obtain exact results.
- Formulae* are ugly but closed form expressions can be obtained

*P. Lansky, F. Polito, LS (2014) The role of detachment of in-links in scale-free networks. J. Phys. A **47**, 345002

Robustness with respect to Detachment (Death)





The presence of detachment kills the scale free behaviour in the critical and sub-critical cases ($\lambda \leq \mu$)

Generalization: Death

Results: Fitting real data

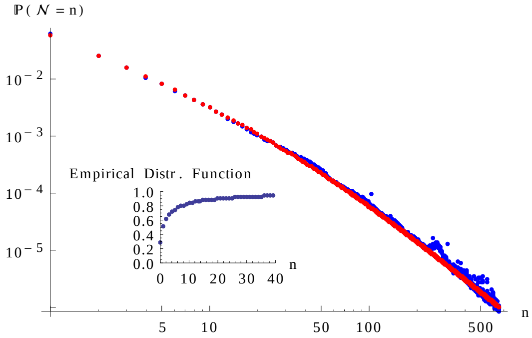


Figure: Fit of the empirical probability mass function for the number of in-links in the WWW. Data are taken from Web Data Commons, University of Manheim. It can be seen that the model (in red) fits the data even for small values of n ($(\lambda; \mu; \beta) = (4; 3.8; 0.272)$). In the inset, the empirical distribution function calculated on the data.

Robustness with respect to fractional linear growth

We used Yule-like models to investigate the case of fractional linear growth

- Finite time: the distribution of the number of in-links for a webpage chosen uniformly at random is rather different from that of the classical Yule model
- When t diverges Scale free appears if $\rho < 1$. We have again the Yule-Simon distribution but the parameter ρ has a new meaning involving the fractionality index $\rho = \frac{\lambda_r}{\beta^\nu}$.

Remark

Any empirical Yule-Simon distribution recorded on real data can be consequence either of an underlying classical Yule model or of a fractional linear Yule model with the same value of ρ.

*P. Lansky, F. Polito, LS (2016) Generalized Nonlinear Yule Models. J. Stat. Phys. **165**, 661-679

Different attachment rules - social networks

The attachment rule is selected to account for two different habits of people joining a group.*

- Each individual can either choose its friend only within the group of its peers (and this typically happens without any specific preference for one of them)$\Rightarrow$ Uniform attachment mechanism (with probability $p \in [0, 1]$)
- Among all nodes of the social $\Rightarrow$ a rich-get-richer mechanism acquires more relevance since old nodes have already a consolidated status in the network network (with a preference for those more popular) (with probability $1 - p \in [0, 1]$)

* A. Pachon, LS and S. Yang (2018) Scale free behavior of networks with the copresence of preferential and uniform attachment rules. Physica D **371**,1-12

Theorem

Consider the UPA model with fixed window's size $l \in \mathbb{N}$, and let $N(k,t)$ be the number of nodes in the network with degree k at the end of period t, and $\bar{P}(k) := \lim_{t\to\infty}[N(k,t)]/t$. Then,

$$\frac{N(k,t)}{t} \to \bar{P}(k) \tag{3}$$

in probability as $t \to \infty$, where for $l = 1$

$$\overline{P}(k) = \begin{cases} \frac{2(1-p)}{3-p}, \ \text{if } k = 1 \\ \frac{(1-p)^2}{(2-p)(3-p)}, \ \text{if } k = 2 \\ \frac{(1-p)^2}{(2-p)(3-p)} B\left(k, 1 + \frac{2}{1-p}\right), \ \text{if } k > 2, \end{cases}$$

and for $l > 1$

Theorem

$$\overline{P}(k) = \begin{cases} \frac{2}{(3-p)}\left(1-\frac{p}{l}\right)^{l}, \ ik\ k=1 \\ \frac{2}{2+k(1-p)}\left(\frac{p}{l}(H_{k-1}-H_k)+\frac{(1-p)(k-1)}{2}\right)\overline{P}(k-1), \ if\ k=2,\ldots, \\ \frac{B\left(k,l+2+\frac{2}{1-p}\right)}{B\left(k+1+\frac{2}{1-p},l+1\right)}\overline{P}(l+1), \ if\ k>l+1, \end{cases}$$

with $B(x,y)$ the Beta function and

$$H_k = \left(\frac{p}{l}\right)^{k-1}\sum_{m=1}^{l-(k-1)}\binom{l-m}{l-m-(k-1)}\left(1-\frac{p}{l}\right)^{l-m-(k-1)}, k \geq 1. \tag{4}$$

Sketch of the proof

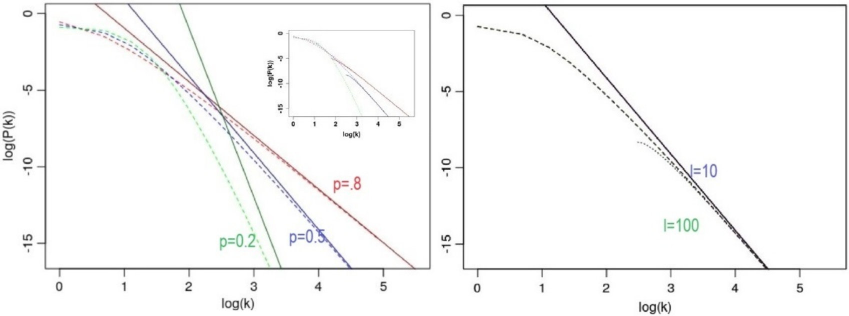
Proof.

The proof includes the following steps:

1. we determine recursively $E[N(k,t)], k = 1, 2, ...;$
2. we prove the existence of $\overline{P(k)} := \lim_{t\to\infty} E[N(t,k)]/t$,
3. we determine an explicit expression for $P(k)$,
4. we use the Azuma–Hoeffding Inequality to prove convergence in probability of $N(k,t)/t$ to $\overline{P(k)}$.

$\square$

Robustness of scale free behaviour for the UPA model



(a) (left panel): The proportion of vertices with degree k versus k is shown in *log–log* scale, for $l = 100$ and different probabilities of window, $p = 0.2$ (green), $p = 0.5$ (blue) and $p = 0.8$ (red). The dashed lines are analytical results, while the solid lines represent the asymptotic values taking only the first term in (18). *Inset:* the same picture but the solid lines here refer to the asymptotic values taking the first and the second term in (18). (b) (right panel): $\log(P(k))$ vs. $\log(k)$. Probability of window $p = 0.5$ and different sizes of the window, $l = 10$ (blue) and $l = 100$ (green). Again dashed lines are analytical results, while the solid lines represent the asymptotic values taking only the first term in (18). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Robustness of scale free behaviour for the UPA model

Remark

1. *The scale free behavior does not disappear by merging preferential and uniform attachment rules.*
2. *Different weights between preferential and uniform attachment rules change the exponent of the power law*
3. *The value of l changes the starting value of t to observe the power law behavior (some connections are "wasted" and do not contribute to the rich-get-richer mechanism.).*

Preferential and anti-preferential attachment

1. Let $Y_t, t = 1, 2, \ldots$ be i.i.d. Bernoulli r.v. of parameter $p \in [0, 1]$ independent of $(G_t), t \geq 1$.
2. Let G_1 be a graph with a single vertex v_1 and m self-loops.
3. For every $t \in N$, construct G_{t+1} from G_t by adding a new vertex v_{t+1} and add m edges between v_{t+1} and vertices of G_t
4. For $1 \leq i \leq t$, independently for each $r \in \{1, \ldots, m\}$, choose the m target vertices in G_t according to the following procedure:
   - If $Y_{t+1} = 0$, we select m random vertices $W^1_{t+1}, \ldots, W^m_{t+1}$ from G_t according to the preferential attachment mechanism

$$\mathbb{P}(W^r_{t+1} = v_i | \mathcal{F}_t) = \frac{d(v_i, t)}{2mt}, \qquad (5)$$

   - If $Y_{t+1} = 1$ we select m random vertices $W^1_{t+1}, \ldots, W^m_{t+1}$ from G_t according to the anti-preferential attachment mechanism

$$\mathbb{P}(W^r_{t+1} = v_i | \mathcal{F}_t) = \frac{2mt + 1 - d(v_i, t)}{t(2mt + 1 - 2m)}. \qquad (6)$$

Results for the PA-APA model

- The anti-preferential mechanism determines a change of the exponent of the degree distribution of the model: the asymptotic degree distribution of such random graph has a right tail decaying as a power-law with exponent $(p-3)/(1-p), p \in [0,1)$.
- The AP-APA model is able to recover any power law with exponent in $(\infty, -3)$ modifying the parameter p.
- In the pure-antipreferential attachment regime the model tends to produce a homogenization of vertices' degree

*U. De Ambroggio, F. Polito, LS (2014) On dynamic random graphs with degree homogenization via anti-preferential attachment probabilities. Physica D: Nonlinear Phenomena **414**

Scale free behaviour in preferential attachment graphs with random initial degree

The scale free behaviour The Barabási–Albert model corresponds to the decaying of its degree distribution as a power law with some characteristic exponent. $\Rightarrow$ concentration results.

- When the number of newly added edges is deterministic and constant the concentration results always hold true
- When the graph process is defined so that, at each integer time t, a new vertex with a random number of edges attached to it, is added to the graph, the existence of asymptotic concentration heavily depends on the distributional properties of the initial degrees themselves.

Robusteness of the scale free behaviour for the PARID model

Parid model:

- In the first step of the process we add two vertices v_0 and v_1 and connect these two vertices with X_1 edges.
- In each subsequent step $t \geq 2$ of the process a new vertex, v_t along with X_t edges are added.
- Each of these X_t edges is generated in the following manner: a vertex v_i, where $i \leq t-1$, is selected with probability

$$\frac{d_i(t-1)}{2\Lambda(t-1)},$$

where $d_i(t-1)$ denotes the degree of vertex v_i after step $t-1$, $\Lambda(t) = \sum_{i=1}^{t} X_i$ and the edge $\{v_i, v_t\}$ is added to the multigraph.

Is the Parid model robust with respect to the scale free property?

The answer changes according with the distribution of the added edges:

- When X is bounded the answer is affirmative *
- When X is regularly-varying of parameter $\alpha = 2$ concentration occurs together with the scale free behaviour
- When X is regularly-varying of parameter $\alpha \in (1, 2)$ no concentration occurs**

*C. Cooper and A. Frieze.(2003) A general model of web graphs. Random Structures Algorithms, **3**,311–335
** T. Makai, F. Polito and LS (2024) Do random initial degrees suppress concentration on preferential attachment graphs? arXiv:2402.04927

Joint works with


Federico Polito


Petr Lansky


Angelica Pachon Pinzon


Shuyi Yang


Umberto De Ambroggio


Tamas Makai

Thank you!