Absence of anomalous dissipation for vortex sheet evolution in 2D incompressible flows

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$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u,$$

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For smooth solutions have

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$$= -\frac{1}{2}\int \operatorname{div}(|u|^2 u) - \int \operatorname{div}(up) - \nu \int |\nabla u|^2$$

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Smooth inviscid flows ($\nu = 0$) conserve kinetic energy

Anomalous dissipation is a cornerstone of turbulence theory:
dissipation rate does not vanish: K41 0^{th} law of turbulence;
inviscid dissipation: related phenomena, inviscid fluid flows ($\nu = 0$) which do not satisfy energy balance;
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Research developed along two fronts: *flexibility* $\times$ *rigidity*

flexibility $\leftrightarrow$ wild solutions $\leftrightarrow$ convex integration

Concentrate on *rigidity* for *2D* flows.

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- **2D result** – Duchon, Robert 2000: initial vorticity in L^p, $p > 3/2$, implies conservation of energy.

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  Kraichnan 2D turbulence theory: postulate forward enstrophy cascade $\to$ regularizing effect in 2D

  Suggests existence of dynamical mechanism preventing anomalous dissipation in 2D even for 'supercritical' flows

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2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$:

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- Smooth vorticity transported in 2D, L^p bounds preserved by evolution
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Note:

- Smooth vorticity transported in 2D, L^p bounds preserved by evolution
- wild solutions: very recent, first example of $\exists$ with control on integrability of vorticity cf. Bruè, Colombo, Kumar, $\omega \in L^p$, $p > 1$, p very close to 1.

Definition

Fix $T > 0$. Let $u_0 \in L^2(\mathbb{T}^2)$ sth $\omega_0 = \operatorname{curl} u_0 \in L^p(\mathbb{T}^2)$, some $p \geq 1$. Say $u \in C(0, T; L^2_{\mathrm{weak}}(\mathbb{T}^2))$ weak solution of incompressible Euler equations, initial data u_0, if $\omega = \operatorname{curl} u \in L^\infty(0, T; L^p(\mathbb{T}^2))$ and

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\operatorname{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

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Existence is known (DiPerna, Majda 87; Vecchi, Wu 93), uniqueness is known in L^∞...and “nearby". Nonuniqueness in L^p, any $2 < p < \infty$, with *nontrivial* rough forcing (Vishik, 2018, see also Albritton, Bruè, Colombo, DeLellis, Giri, Janisch, Kwon, 2021). Recent work: nonuniqueness in L^p, no forcing, $p > 1$, p close to 1 (Bruè, Colombo, Kumar, 2024).

Vanishing viscosity solutions

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Let $u \in C(0, T; L^2_w(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;
2. there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that, as $\nu \to 0$,
    - $u^\nu \rightharpoonup u$ weakly* in $L^\infty(0, T; L^2(\mathbb{T}^2))$;
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The family $\{u^\nu\}$ is called a *physical realization* of u.

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Theorem (Cheskidov,Lopes Filho, NL, Shvydkoy 2016)

Let $u \in C(0, T; L^2_w(\mathbb{T}^2))$ be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that $u_0 \in L^2$ is such that $\operatorname{curl} u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$, for some $p > 1$. Suppose that there is a physical realization $\{u^\nu\}$ such that $\{\omega_0^\nu\}$ is bounded in $L^p(\mathbb{T}^2)$. Then u conserves energy.

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Obs. $1 < p < 3/2$ 'supercritical'.

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Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial.

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$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \sup_\nu \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$.

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Energy methods plus Gagliardo-Nirenberg, plus maximum principle for L^p-norm of vorticity give:

$$y' \leq -2C_0 \nu \, y^{\frac{2}{2-p}} \text{ and } \frac{2}{2-p} > 2.$$

Discussion of proof:
Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial.
u is physically realizable $\Longrightarrow \exists$ physical realization $\{u^\nu\}$ solutions of Navier-Stokes with $\{\omega_0^\nu\}$ bounded in L^p. $\omega^\nu = \mathrm{curl}\, u^\nu$.
Vorticity equation:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \sup_\nu \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$.
Energy methods plus Gagliardo-Nirenberg, plus maximum principle for L^p-norm of vorticity give:

$$y' \leq -2C_0 \nu \, y^{\frac{2}{2-p}} \text{ and } \frac{2}{2-p} > 2.$$

Integrate in time, use $\|\omega_0^\nu\|_{L^2} \to +\infty$ and substitute $y(t) = \|\omega^\nu\|_{L^2}^2$ to get

$$\|\omega^\nu(t,\cdot)\|_{L^2}^2 \leq (o(1) + C\nu t)^{-\frac{2-p}{p}}.$$

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{1}$$

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Integrate in time and use the estimate for vorticity to get

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -2\nu\int_0^t (o(1) + C\nu s)^{-\frac{2-p}{p}}\, ds$$

$$= -C\left[\left(o(1) + \widetilde{C}\nu t\right)^{\frac{2(p-1)}{p}} - o(1)\right] \gtrsim -(\nu t)^{\frac{2(p-1)}{p}} + o(1).$$

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Now $p > 1 \implies$ RHS$\to 0$ as $\nu \to 0$. Therefore:

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Strong convergence of initial data $\implies \|u(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$ as desired.

Key steps in the proof:

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Show dissipation vanishes:

$$\nu \int_0^T \|\omega^\nu(t,\cdot)\|^2 \, dt \to 0 \text{ as } \nu \to 0 :$$

$$\nu \int_0^T \|\omega^\nu(t,\cdot)\|^2 \, dt \lesssim \nu^{2(p-1)/p} T^{2(p-1)/p},$$

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Then use strong convergence in L^2 of physical realizations:

$$\|u(t,\cdot)\|_{L^2}^2 - \|u_0\|_{L^2}^2 = \lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

Surprisingly, vanishing dissipation is a *consequence* of strong convergence. In fact, it holds that:

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Theorem (Lanthaler, Mishra, Parés-Pulido 2021)

Suppose $u_0 \in L^2(\mathbb{T}^2)$ *and* u *is* physically realizable *weak solution with* $u(0) = u_0$. *Let* u^ν physical realization: $u^\nu \rightharpoonup u$; $u^\nu(0) \to u_0$. *Then the following are equivalent:*

1. $u^\nu \to u$ *strongly in* $L^p((0,T); L^2(\mathbb{T}^2))$, *some* $1 \leq p < \infty$;
2. u *conservative weak solution.*

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1. $u^\nu \to u$ *strongly in* $L^p((0,T); L^2(\mathbb{T}^2))$*, some* $1 \leq p < \infty$*;*
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In particular, it was established that:

$$\text{Strong convergence in } L^2 \implies \text{ no anomalous dissipation,}$$

$$\text{Strong convergence in } L^2 + \text{ no anomalous dissipation}$$

$$\implies \text{ no inviscid dissipation.}$$

Proof of 'no inviscid dissipation $\Longrightarrow$ strong convergence'

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For any $0 \leq t \leq T$, have

$$\begin{aligned} T\|u(t)\|_{L^2}^2 &= \int_0^T \|u(s)\|_{L^2}^2 \leq \liminf_{\nu\to 0} \int_0^T \|u^\nu(s)\|_{L^2}^2 \\ &\leq \limsup_{\nu\to 0} \int_0^T \|u^\nu(s)\|_{L^2}^2 \leq \limsup_{\nu\to 0} T\|u^\nu(0)\|_{L^2}^2 \\ &= T\|u(0)\|_{L^2}^2. \end{aligned}$$

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Therefore all $\leq$ are $=$ and have convergence of norms.

Convergence of norms $+$ weak convergence $\Longrightarrow$ strong convergence.

Proof of strong convergence $\implies$ 'no inviscid dissipation': the heart of the matter

Proof of strong convergence $\Longrightarrow$ 'no inviscid dissipation': the heart of the matter

The key lemma is:

Lemma

Let $\{u^\nu\}_{\nu>0}$ *precompact in* $L^2(0,T;L^2(\mathbb{T}^2))$*, div-free. Then* $\exists$

$$\sigma : [0,\infty) \to [0,\infty) \text{ such that } \lim_{z\to\infty} \sigma(z) = \infty$$

so that, for every $0 < \delta < t < T$

$$\left(\int_\delta^t \|\omega^\nu(s)\|_{L^2}^2 \, \mathrm{d}s\right)^2 \sigma\left(\int_\delta^t \|\omega^\nu(s)\|_{L^2}^2 \, \mathrm{d}s\right) \leq \int_\delta^t \|\nabla\omega^\nu(s)\|_{L^2}^2 \, \mathrm{d}s.$$

The lemma allows to obtain a differential inequality for

$$y_\delta^\nu = y_\delta^\nu(t) := \nu \int_\delta^t \|\omega^\nu(s)\|^2 \, \mathrm{d}s,$$

namely

$$(\star) \quad \frac{d}{dt} y_\delta^\nu \leq M_\delta - (y_\delta^\nu)^2 \, \sigma\left(\frac{y_\delta^\nu}{\nu}\right).$$

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We then show

Lemma

Let $y_\delta^\nu \in W^{1,1}([0,T])$ increasing functions sastisfying $(\star)$. Then

$$\limsup_{\nu \to 0} y_\delta^\nu(T) = 0.$$

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Strong convergence of initial data + continuity in time of weak Euler solution allows $\delta \to 0$.

OBS.

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Re-write key lemma as:

$$\exists\, \Upsilon = \Upsilon(z) \text{ superquadratic, } \lim_{|z|\to\infty} \frac{\Upsilon(z)}{z^2} = +\infty, \text{ such that:}$$

$\{u^\nu\}_{\nu>0}$ precompact in $L^2(0,T;L^2(\mathbb{T}^2))$, div-free. Then, for every $0 < \delta < t < T$

$$\Upsilon\left(\int_\delta^t \|\omega^\nu(s)\|_{L^2}^2\,\mathrm{d}s\right) \leq \int_\delta^t \|\nabla\omega^\nu(s)\|_{L^2}^2\,\mathrm{d}s.$$

Consequences of LMPP 2021:

$u_0 \in L^2$, $\omega_0 \in X$, $X \Subset H^{-1}(\mathbb{T}^2)$ (compact imbedding!), ω^ν bounded in $L^\infty_t(X)$. Then any weak limit of u^ν is energy conservative.

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Examples: L^p, $p > 1$; $L(\log L)^\alpha$, $\alpha > 1/2$; $L^{(1,q)}$, $1 \leq q < 2$. All of these are rearrangement invariant subspaces of L^1 compactly imbedded in H^{-1}. (Cf. H^{-1}-stability, LNT 2000.)

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What happens in $L^1 \cap H^{-1}$? Or $\mathcal{BM} \cap H^{-1}$? Delort 1991/Vecchi, Wu 1993: exists weak solution, no concentrations *in vorticity*. Estimates do not exclude concentrations in *energy*, *i.e.* energy defect.

Vorticity in $\mathcal{BM} \cap H^{-1}$: vortex sheets.

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Vortex sheets are ubiquitous phenomena in ideal (or high Reynolds number) flows – e.g. they appear in flow past an obstacle (for instance, an airfoil – vortex sheets trailing an airplane wing), mixing of fluids with different densities, etc.

Vortex sheet flows arise, largely, as approximate models for nearly inviscid flows past an obstacle. A ‘good’ approximation for the flow is an irrotational velocity field (aka potential flow) except in a thin layer close to the obstacle, where there are large gradients of velocity, i.e., where vorticity is concentrated. The creation of this shear layer is due to the effect of (small) viscosity on the fluid-structure interaction.

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Substitute shear layer by vortex sheet and set viscosity $= 0$.

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We may consider more general flows, for which

$$\omega \in \mathcal{BM} \cap H^{-1},$$

i.e., u has (locally) bounded kinetic energy (exclude point vortices).

No compactness and no anomalous dissipation

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What can be done for $\omega_0 \in L^1(\mathbb{T}^2) \cap H^{-1}(\mathbb{T}^2)$ or $\mathcal{BM}(\mathbb{T}^2) \cap H^{-1}(\mathbb{T}^2)$?

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These spaces are not compact in $H^{-1}(\mathbb{T}^2)$, therefore no *a priori* strong convergence of u^ν in $L^2(0, T; L^2(\mathbb{T}^2))$.

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Do not expect energy balance – inviscid dissipation is possible.

Instead, investigate anomalous dissipation.

Consider solutions of NS whose initial data converges, at least weakly, in $L^2(\mathbb{T}^2)$.

Theorem (de Rosa, Park 2024; also Elgindi, Lopes Filho, NL 2024)

Assume

1. $u_0^\nu \to u_0$ *strong*-$L^2(\mathbb{T}^2)$,
2. ω^ν *bounded in* $L^\infty(0, T; L^1(\mathbb{T}^2))$,
3. $F^\nu \rightharpoonup F$ *weak*-$L^2(0, T; L^2(\mathbb{T}^2))$,
4. $\operatorname{curl} F^\nu(\cdot, t) \rightharpoonup \operatorname{curl} F(\cdot, t)$ *weak*-$L^1(\mathbb{T}^2)$, *a.e.* t,
5. $\int \sup_\nu \|\operatorname{curl} F^\nu(\cdot, t)\|_{L^1} \, \mathrm{d}t < \infty$.

If

$$\sup_\nu \sup_t \sup_z \int_{\{|x-z|<r\}} |\omega^\nu| \, \mathrm{d}x \to 0 \text{ as } r \to 0$$

then

$$\limsup_{\nu \to 0} \; \nu \int_0^T \|\omega^\nu(\cdot, t)\|_{L^2}^2 \, \mathrm{d}t = 0.$$

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Under assumptions in Theorem any weak limit is a physically realizable weak solution of 2D Euler, with $u(0) = u_0$.

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Recall Nash's inequality:

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We begin with a refinement:

Proposition

Let $\mathcal{F} \subset L^1(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$ *such that*

- A $\|f\|_{L^1} \leq K$, *for all* $f \in \mathcal{F}$;
- B $\lim_{r\to 0} \sup_{f\in\mathcal{F}} \sup_z \int_{\{|x-z|<r\}} |f(x)|\,dx = 0.$

Then $\exists \Upsilon \in \mathcal{C}^1$, *convex, increasing, superquadratic such that,* $\forall f \in \mathcal{F}$,

$$\Upsilon(\|f\|_{L^2}^2) \leq \|\nabla f\|_{L^2}^2$$

Obs. $\Upsilon = \Upsilon_K$. By construction $\Upsilon = (\Phi^{-1})^2$, Φ concave, increasing, sublinear.

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Discussion of proof of Theorem, ELN 2024 version:
Use refinement of Nash for

$$\mathcal{F} = \{\omega^\nu(\cdot, t),\ 0 < t < T,\ \nu > 0\}.$$

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Set $\zeta_\delta^\nu = \zeta_\delta^\nu(t) := \nu \int_\delta^t \|\omega^\nu(s)\|_{L^2}^2 \, \mathrm{d}s$. Then ζ_δ^ν satisfies the differential inequality

$$\frac{d}{dt}\zeta_\delta^\nu \leq M_\delta - \nu^2(t-\delta)\Upsilon\left(\frac{\zeta_\delta^\nu}{\nu(t-\delta)}\right).$$

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But $d\zeta_\delta^\nu/dt = \nu\|\omega^\nu(t)\|^2 \geq 0$. Therefore

$$\nu^2(t-\delta)\Upsilon\left(\frac{\zeta_\delta^\nu(t)}{\nu(t-\delta)}\right) \leq M_\delta.$$

$$\Longrightarrow \zeta_\delta^\nu(t) \leq \nu(t-\delta)\Phi\left(\frac{\sqrt{M_\delta}}{\nu\sqrt{t-\delta}}\right)$$

Letting

$$X_\nu := \frac{\sqrt{M_\delta}}{\nu\sqrt{T-\delta}}$$

we get

$$\boxed{\zeta_\delta^\nu(T) \leq \sqrt{M_\delta}\sqrt{T-\delta}\,\frac{\Phi(X_\nu)}{X_\nu}}.$$

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This limit vanishes since $u_0^\nu \to u_0$ strong-$L^2(\mathbb{T}^2)$ and weak limits of u^ν are weak 2D Euler solutions which, in turn, are right-continuous in time into L^2 at $t=0$.

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$$\vartheta(r) = \frac{1}{K} \sup_{f\in\mathcal{F}} \sup_{z\in\mathbb{T}^2} \int_{\{|z-y|<r\}} |f(y)| \, \mathrm{d}y. \tag{2}$$

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Set

$$\eta(r) = \max\left\{\vartheta(r), \frac{r}{\pi}\right\}, \quad \text{for } 0 \leq r \leq \pi \tag{3}$$

and

$$\overline{\eta} = \overline{\eta}(r) = \begin{cases} \eta(r) & \text{if } 0 \leq r \leq \pi, \\ 1 & \text{if } r > \pi. \end{cases} \tag{4}$$

In the refinement of Nash's inequality $\Upsilon = (\Phi^{-1})^2$. The concave, increasing, sublinear and $\mathcal{C}^1$ function Φ has a very explicit form:

Recall $\|f\|_{L^1} \leq K$, $f \in \mathcal{F}$. Introduce

$$\vartheta(r) = \frac{1}{K} \sup_{f\in\mathcal{F}} \sup_{z\in\mathbb{T}^2} \int_{\{|z-y|<r\}} |f(y)| \, \mathrm{d}y. \tag{2}$$

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Then we can use

$$\Phi = \Phi(x) = \int_0^x C\sqrt{\overline{\eta}\left(\pi y^{-1/4}\right)} \, \mathrm{d}y, \text{ for } x \geq 0.$$

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Then, for large x, we can use

$$\Phi = \Phi(x) = C\frac{x}{\sqrt[4]{\log x}}.$$

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Let ω_0^ν be a bounded sequence in $\mathcal{BM}(\mathbb{T}^2) \cap H^{-1}(\mathbb{T}^2)$. Suppose $\omega_0^\nu = \mu_0^\nu + w_0^\nu$, with $\mu_0^\nu \in \mathcal{BM}(\mathbb{T}^2)$, $\mu_0 \geq 0$, and $w_0^\nu \in L^1(\mathbb{T}^2)$ and $w_0^\nu \rightharpoonup w$ weak-$L^1(\mathbb{T}^2)$. Then there is no anomalous dissipation for u^ν.

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Obs. Not necessarily sharp.

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Let $\varphi \in C_c^\infty(0, +\infty)$. Assume

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Orbita Mathematicae is the new journal of the Unión Matemática de América Latina y el Caribe (UMALCA).

Orbita in Classical Latin originally meant, “the track left by a wheel or a way well-worn by one's ancestors”. The Renaissance brought another meaning to this word: “the path of a celestial body when it revolves around another body due to their mutual gravitational attraction”. Finally, modern times added yet another meaning to the Latin *Orbita*: “the sphere of influence”. The title of the journal reflects the historical evolution of mathematics. All sciences are understood, nowadays, to belong to the *Orbita Mathematicae*.

The journal *Orbita Mathematicae* publishes original research articles of the highest level and aims to position itself among the most prestigious international mathematics journals.

Figure: *Orbita Mathematicae*: a new UMALCA journal

Thank you

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Grazie