Comparison results for elliptic equations via Steiner symmetrization

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Model problem

$$\begin{cases} -\Delta_{p,x}\, u - u_{yy} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Delta_{p,x}\, u = \mathrm{div}(|\nabla_x u|^{p-2}\nabla_x u)$$

$\Omega = \Omega_1 \times (0,1)\,, \quad \Omega_1 \subset \mathbb{R}^n$ open bounded Lipschitz domain

$$f \geq 0 \qquad f \in L^q(\Omega), \qquad q = \max\{p, 2\}$$

- F.Brock, I.Diaz, A.Ferone, D.Gomez, A.M., Ann.I.H. Poincaré, 2021
- I.Diaz, A.Ferone, A.M., J. Math. Anal. Appl. 2024

Comparison result via Schwarz symmetrization

$u \in W_0^{1,2}(\Omega)$ and $v \in W_0^{1,2}(\Omega^\star)$ weak solutions to

$$\begin{cases} -\text{div}\,(A(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star\,, \\ v = 0 & \text{on } \partial\Omega^\star\,. \end{cases}$$

Ω bounded domain,

$$A(x) = (a_{ij}(x))_{ij} \qquad a_{ij} \in L^\infty(\Omega) \qquad a_{ij}(x)\xi_i\xi_j \geq |\xi|^2\,,$$

$$f \in L^q(\Omega), \qquad q \geq (2^*)'$$

$$\Downarrow$$

$$u^\star(z) \leq v(z) \qquad \text{a.e. } z \in \Omega^\star$$

- Talenti, 1976

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$$\|u(z)\| = \|u^\star(z)\|$$

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- Talenti, 1976

A few references

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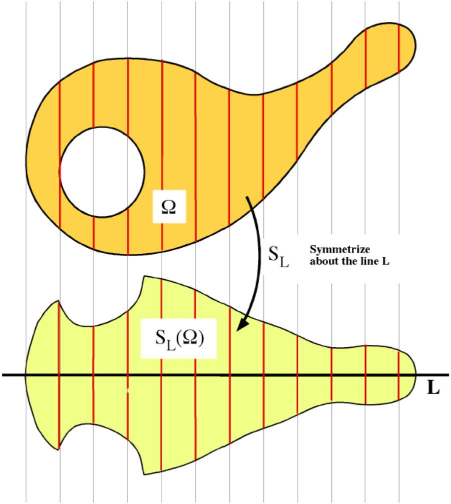
Steiner rearrangement of a set



Steiner rearrangement of a function

$$N \geq 2\,, \qquad z \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$$

$$z = (x, y), \qquad x \in \mathbb{R}^n, \qquad y \in \mathbb{R}^m$$

$$\Omega_y := \{x \in \mathbb{R}^n : (x, y) \in \Omega\}\,, \quad y \in \mathbb{R}^m\,.$$

$\Omega^\#$ Steiner rearrangement of Ω

$$x \in \Omega_y \to u(x, y) \in \mathbb{R}$$

$$u^\#(x, y) = (u(\cdot, y))^\star = u^*(\omega_n |x|^n, y) \qquad (x, y) \in \Omega^\#\,.$$

$u^\#$ Steiner rearrangement of u

Comparison result via Steiner symmetrization for linear elliptic operators

$u \in W_0^{1,2}(\Omega)$ and $v \in W_0^{1,2}(\Omega^\#)$ weak solutions to respectively

$$\begin{cases} -\text{div}\,(A(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} -\Delta v = f^\# & \text{in } \Omega^\#\,, \\ v = 0 & \text{on } \partial\Omega^\#\,. \end{cases}$$

Ω bounded domain, $\quad f \in L^q(\Omega)$, $q > \frac{N}{2}$,

$$\Downarrow$$

$$\int_{B_r(0)} u^\#(x,y)dx \le \int_{B_r(0)} v(x,y)dx \quad r \ge 0, \text{for a.e. } y \in \mathbb{R}^m$$

- A.Alvino, I.Diaz, P-L.Lions, G.Trombetti, 1996

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Proof of ADLT of Comparison result via Steiner symmetrization: main tools

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- Integration on the level sets of $u(\cdot, y)$:

$$\int_{\{x\colon\, u(\cdot,y)>t\}} \frac{\partial^2 u}{\partial x^2}\, dx + \int_{\{x\colon\, u(\cdot,y)>t\}} \frac{\partial^2 u}{\partial y^2}\, dx = \int_{\{x\colon\, u(\cdot,y)>t\}} f\, dx$$

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- Second order derivation formula

$$\int_{\{x:\ u(\cdot,y)>t\}} \frac{\partial^2 u}{\partial x^2}\, dx = \frac{\partial^2}{\partial y^2} \int_{\{x:\ u(\cdot,y)>t\}} u\, dx + \text{Reminder term}$$

A.Alvino, I.Diaz, P-L.Lions, G.Trombetti, 1996,
V.Ferone- A.M., 1998

Proof of ADLT of Comparison result via Steiner symmetrization: main tools

- Integration on the level sets of $u(\cdot, y)$ and derivation formula give:

$$-\int_{\{x:\ u(\cdot,y)>t\}} \frac{\partial^2 u}{\partial x^2}\, dx - \frac{\partial^2}{\partial y^2} \int_{\{x:\ u(\cdot,y)>t\}} u\, dx = \int_{\{x:\ u(\cdot,y)>t\}} f\, dx$$

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$$U(y, s) = \int_0^s u^*(\sigma, y)\, d\sigma$$

$$-n^2 \omega_n^{\frac{2}{n}} s^{2-2/n} \frac{\partial^2 U}{\partial s^2} - \frac{\partial^2 U}{\partial y^2} \le \int_0^s f^*(\sigma, y) d\sigma$$

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- Maximum principle

$$U(y,s) = \int_0^s u^*(\sigma,y)\,d\sigma \le V(y,s) = \int_0^s v^*(\sigma,y)\,d\sigma$$

Comparison result via Steiner symmetrization: a different approach

$u \in W_0^{1,2}(\Omega)$ and $v \in W_0^{1,2}(\Omega^\#)$ weak solutions to respectively

$$\begin{cases} -\Delta\, u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} -\Delta v = f^\# & \text{in } \Omega^\#\,, \\ v = 0 & \text{on } \partial\Omega^\#\,. \end{cases}$$

Ω bounded domain, $\quad f \in L^q(\Omega)$, $q > \frac{N}{2}$,

$$\Downarrow$$

$$\int_{B_r(0)} u^\#(x,y)dx \le \int_{B_r(0)} v(x,y)dx \qquad r \ge 0, \text{for a.e. } y \in \mathbb{R}^m$$

- F. Brock, F. Chiacchio, A. Ferone, A.M., Adv. Math. 2018

A different approach: discretization of gradient and Laplace operator

Discretization of the gradients + Riesz Inequality + Hardy-Littelwood equality

$$\int_{\mathbb{R}^N} \nabla_x u(z) \cdot \nabla_x w(z)\, dz$$

$$= \frac{C}{n} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{B_1(0)} \frac{u(x+\epsilon h, y) - u(x,y)}{\epsilon} \frac{w(x+\epsilon h, y) - w(x,y)}{\epsilon} \phi(h) dh dz$$

$$\geq \frac{C}{n} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{B_1(0)} \frac{u^{\#}(x+\epsilon h, y) - u^{\#}(x,y)}{\epsilon} \frac{w^{\#}(x+\epsilon h, y) - w^{\#}(x,y)}{\epsilon} \phi(h) dh dz$$

$$= \int_{\mathbb{R}^N} \nabla u^{\#}(z) \cdot \nabla w^{\#}(z)\, dz$$

A different approach: discretization of gradient and Laplace operator

- Inequalities involving Laplacian operator

$$\int_{\mathbb{R}^N} \nabla u(z) \cdot \nabla w(z) dz \geq \int_{\mathbb{R}^N} \nabla u^\#(z) \cdot \nabla w^\#(z) dz \,.$$

$$\Downarrow$$

$$-\int_\Omega \Delta_x u(x,y) w(x,y) dx dy \geq \int_{\Omega^\#} \nabla_x u^\#(x,y) \cdot \nabla_x w^\#(x,y) dx dy$$

$$-\int_\Omega u_{y_i y_i}(x,y) w(x,y) dx dy \geq \int_{\Omega^\#} u^\#_{y_i}(x,y) \cdot w^\#_{y_i}(x,y) dx dy$$

where $u \geq 0$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $u = 0$ on $\partial\Omega$, $w^\# \in W^{1,\infty}$

A more general Pólya-Szegő inequalities

Ω be a bounded domain of $\mathbb{R}^n$, $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$,
$W : (0, |\Omega|_n)] \to \mathbb{R}$ be a nonincreasing function belonging to $W^{1,p}(a, \mathcal{L}^n(\Omega))$ for every $a > 0$, such that $W(|O|_n)) = 0$

$$-W'(s) \leq C(-u^*)'(s) \qquad for\ a.e.\ s \in (0, |\Omega|_n)),$$

$$\Downarrow$$

$w \in W_0^{1,p}(\Omega)$ is the unique function satisfying $w^\star = W^\star$,

$$\int_{\mathbb{R}^N} u(z)w(z)dz = \int_{\mathbb{R}^N} u^\star(z)w^\star(z)dz$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx \geq \int_{\Omega^\star} |\nabla u^\star|^{p-2}\nabla u^\star \cdot \nabla w^\star\, dx\,.$$

Nonlinear comparison result for Schwarz symmetrization

Let $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,p}(\Omega^\star)$ be weak solutions to the problems respectively

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \quad \begin{cases} -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) = f^\star & \text{in } \Omega^\star\,, \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}$$

$$\Downarrow$$

$$u^\star(x) \leq v(x) \quad \text{for a.e. } x \in \Omega\,.$$

- Talenti, 1979
- F. Brock, F. Chiacchio, A. Ferone, A.M., 2018, for a different proof

$$\begin{cases} -\operatorname{div}_x \Big(a(|\nabla_x u|)\nabla_x u \Big) - u_{yy} = f & \text{in } \Omega = \Omega_1 \times (0,1) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\Omega_1 \subset \mathbb{R}^n$ open bounded Lipschitz $\qquad f \in L^{\max\{2,p'\}}(\Omega)$

- $a : (0,+\infty) \to (0,+\infty)$ $\quad \mathcal{C}^1$ function ,
- $t^{p-2} \le a(t) \le Ct^{p-2} \quad p > 1,$
- $-1 < i_a \le s_a < \infty$,

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)}, \qquad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}$$

- I.Diaz, A.Ferone, A.M., J. Math. Anal. Appl. 2024

Comparison result for anisotropic quasilinear equations: smooth case

$$u \in X^p(\Omega) = \{u \in W_0^{1,1}(\Omega) : |\nabla_x u| \in L^p(\Omega), |\nabla_y u| \in L^2(\Omega)\}$$

and

$$v \in X^p(\Omega^\#) = \{u \in W_0^{1,1}(\Omega^\#) : |\nabla_x u| \in L^p(\Omega^\#), |\nabla_y u| \in L^2(\Omega^\#)\}$$

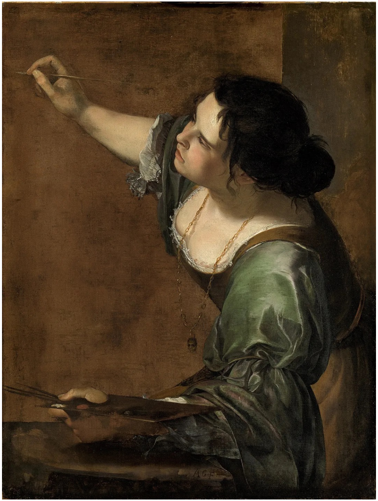
weak solutions to respectively

$$\begin{cases} -\operatorname{div}_x \Big(a(|\nabla_x u|)\nabla_x u \Big) - u_{yy} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \begin{cases} -\Delta_{p,x}\, v - v_{yy} = f^\# & \text{in} \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\# \end{cases}$$

$$\Omega = \Omega_1 \times (0,1)\,, \Omega_1 \subset \mathbb{R}^n \text{ open bounded Lipschitz}$$

$$\Downarrow$$

$$\int_{B_r(0)} u^\#(x,y)dx \leq \int_{B_r(0)} v(x,y)dx \quad r \geq 0, \text{for a.e. } y \in (0,1)$$



Dedicated to all

Women in Mathematics:

«Mostrerò alla
Vostra Illustre Signoria
ciò che una donna può fare»

Artemisia Gentileschi, 1593-1656