On the quantitative isoperimetric inequality

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The isoperimetric inequality

Theorem

If E is a Lebesgue measurable set with the same measure of a ball B, then

$$P(E) \geq P(B) \,.$$

The equality holds if and only if E is a ball.

Some proofs in the two-dimensional case:

1. Zenodorus (c. 200 - c. 140 BC): polygons
2. Steiner (1838): necessary conditions for the existence
3. Weierstrass (1870): first rigorous proof
4. Hurwitz (1901): Wirtinger's inequality

Some proofs in the N-dimensional case:

1. Brunn-Minkowski (1987-1996): Minkowski content
2. Aleksandrov (1958): boundary with constant mean curvature
3. De Giorgi (1958): symmetrization
4. Gromov (1986): mass transportation
5. Cabré (2000): solutions of the Neumann problem for the laplacian

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Can we estimate the distance from the ball if we know $P(E) - P(B)$?

The isoperimetric deficit is defined as

$$\delta(\Omega) = \frac{P(\Omega) - P(B)}{P(B)}\,, \quad |B| = |\Omega|\,,\; B = \text{ball}$$

The quantitative isoperimetric inequalities

[Distance $(\Omega$, ball of same measure$)]^p \leq C(n)\delta(\Omega)$ (or a function of δ)

Some distances

1. $\dfrac{d_{\mathcal{H}}(\Omega, B_G)}{r}\,, |B_G| = |\Omega| = \omega_n r^n\,,\quad G = \text{barycentre of } \Omega$
   ("uniform spherical deviation")[a]

2. $\lambda(\Omega) = \min\limits_{x\in\mathbb{R}^N}\left\{\dfrac{|\Omega\Delta B_x|}{|\Omega|}\,, |B_x| = |\Omega|\right\}$
   ("Fraenkel asymmetry")[b]

3. $\lambda_0(\Omega) = \dfrac{|\Omega\Delta B_G|}{|\Omega|}\,,\ |B_G| = |\Omega|\,,\quad G = \text{barycentre of } \Omega$
   ("barycentric asymmetry")[c]

4. $\lambda_{\mathcal{H}}(\Omega) = \min\limits_{x\in\mathbb{R}^N}\left\{\dfrac{d_{\mathcal{H}}(\Omega, B_x)}{r}\,, |B_x| = |\Omega| = \omega_n r^n\right\}$
   ("deviation from the spherical shape")[d]

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[a] Fuglede 1989, convex sets, nearly spherical sets

[b] Hall, Hayman, Weitsman 1991, Fusco, Maggi, Pratelli 2008, Figalli, Maggi, Pratelli 2010, Cicalese, Leonardi 2012, Dambrine, Lamboley, 2018

[c] Fuglede 1993, convex sets

[d] Fusco, Gelli, Pisante, 2011

A first quantitative isoperimetric inequality, with the Fraenkel asymmetry:

$$\lambda(\Omega) = \min_{x \in \mathbb{R}^N} \left\{ \frac{|\Omega \Delta B_x|}{|\Omega|}, |B_x| = |\Omega| \right\}$$

Theorem (Fusco, Maggi and Pratelli, 2008)

There exists $C_N > 0$ such that for every Ω, $\lambda(\Omega)^2 \leq C_N \, \delta(\Omega)$.

What is the value of the optimal constant C_N?

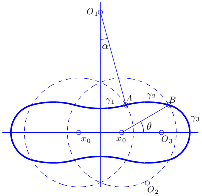
Theorem (Campi 1992, Alvino, Ferone and Nitsch 2011)

$$\inf_{\Omega \, convex \subset \mathbb{R}^2 \neq B} \frac{\delta(\Omega)}{\lambda(\Omega)^2} \approx 0.406;$$

the infimum is realized by an explicitely described stadium.

Some estimates of $\inf_{\Omega \subset \mathbb{R}^2} \frac{\delta(\Omega)}{\lambda^2(\Omega)} = I$

- Hall-Hayman-Weitsman: $I \geq 0.02$
- Figalli-Maggi-Pratelli: $\inf_{\Omega \subset \mathbb{R}^n} \frac{\delta(\Omega)}{\lambda^2(\Omega)} \geq \frac{(2 - 2^{\frac{1}{n'}})^3}{181^2 \cdot n^{14}}$ therefore $I \geq 3.7 \times 10^{-10}$
- Zhao-Ding-Jiang : $I \geq 0.0625$
- Conjecture (Cicalese, Leonardi; Bianchini, C., Henrot): $I \approx 0.393$, realized by an explicitely described mask[1]



[1]domain with regular boundary, composed by arcs of circle, with 2 symmetry axes and 2 "Fraenkel balls"

Theorem (C. Bianchini, G.C. and A. Henrot)

There exists a set $\Omega^* \neq B$ which minimizes $\frac{\delta(\Omega)}{\lambda^2(\Omega)}$ among all the subsets of $\mathbb{R}^2$.

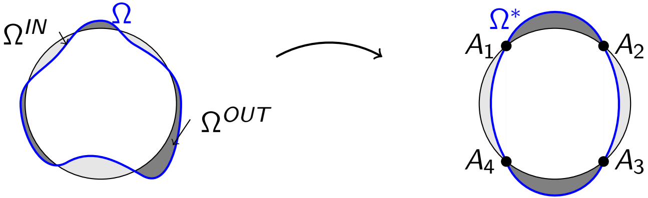
- Ω^* has at least two optimal balls for the Fraenkel asymmetry.
- Ω^* is not convex and has at most six connected components.

N.B.: Cicalese and Leonardi had shown that Ω^* is $C^{1,1}$ and its boundary is composed of arcs of circle. In any connected component of $\mathbb{R}^2 \setminus \cup_{x \in Z(\Omega^*)}(x + \partial B)$, $\partial\Omega^*$ is a union of arcs of circle with the same radius ($Z(\Omega^*)$: set of the centers of the optimal balls).

N.B.: Our result gives a new proof of the quantitative isoperimetric inequality in the plane!

Idea of our proof of the existence of a minimizer: sequences Ω_n converging to the ball

We symmetrize each set Ω of the sequence in this way:



Ω^* is obtained by distributing half of the external matter on the north pole and half on the south pole, while half of the internal matter is put on the west pole and half on the east pole, preserving the total lengths of $\partial\Omega^{OUT} \cap \partial B$ and $\partial\Omega^{IN} \cap \partial B$.
$\partial\Omega^*$ by composed of arcs of circle.

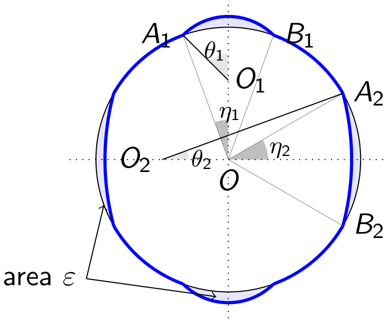
N.B.: Ω^* is well defined if $|\Omega \Delta B|$ is small.

Exclusion of sequences converging to the ball

Theorem

Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$, be a sequence of sets, such that $|\Omega_\varepsilon| = \pi = |B|$ where B is a unit ball. Assume that $|B\Delta\Omega_\varepsilon| = \dfrac{4\varepsilon}{\pi}$. Then

$$\liminf_{\varepsilon\to0} \frac{\delta(\Omega_\varepsilon^*)}{\lambda^2(\Omega_\varepsilon^*)} \geq \frac{\pi}{8(4-\pi)}.$$



Exclusion of sequences converging to the ball

This rearrangement decreases (asymptotically) $\dfrac{\delta(\Omega)}{\lambda^2(\Omega)}$:

Corollary

Let $\varepsilon > 0$. Let Ω_ε be a sequence of sets converging to a ball B such that $|B\Delta\Omega_\varepsilon| = \frac{4\varepsilon}{\pi}$. Then

$$\liminf_{\varepsilon\to 0} \frac{\delta(\Omega_\varepsilon)}{\lambda^2(\Omega_\varepsilon)} \geq \frac{\pi}{8(4-\pi)} \approx 0.46$$

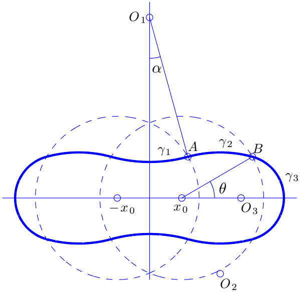
Recall that $\inf_{\Omega convex \subset \mathbb{R}^2 \neq B} \dfrac{\delta(\Omega)}{\lambda(\Omega)^2} \approx 0.41$

IF one is able to prove that

- the optimal set Ω^* has two perpendicular axes of symmetry;
- the optimal set has exactly two optimal balls B_1 and B_2 realizing the Fraenkel asymmetry,

then he is led to solve a minimization problem in finite dimension.

Therefore one can also compute $\inf_{\Omega} \dfrac{\delta(\Omega)}{\lambda^2(\Omega)}$...



A quantitative isoperimetric inequality with the barycentric asymmetry:

$$\lambda_0(\Omega) = \frac{|\Omega \Delta B_G|}{|\Omega|}\,,\ |B_G| = |\Omega|\,,\quad G = \text{barycentre of }\ \Omega$$

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N.B.: $\lambda_0(\Omega) \geq \lambda(\Omega)$ and therefore $\dfrac{\delta(\Omega)}{\lambda^2(\Omega)} \geq \dfrac{\delta(\Omega)}{\lambda_0^2(\Omega)}$.

Can we compute $\inf\limits_{\Omega} \dfrac{\delta(\Omega)}{\lambda_0^2(\Omega)}$ and then get an estimate from below of $\dfrac{\delta(\Omega)}{\lambda^2(\Omega)}$?

N.B.: Estimate from above: $\inf\limits_{\Omega} \dfrac{\delta(\Omega)}{\lambda^2(\Omega)} \leq \dfrac{\delta(\text{mask})}{\lambda^2(\text{mask})} \approx 0.393$

The quantitative isoperimetric inequality with λ_0

Problem: study $\delta(\Omega) \geq C[\lambda_0(\Omega)]^2$ in the plane

Existence of a constant C:

- we deduce from Fuglede (1989) : nearly spherical sets (star-shaped sets with respect to their barycenter, which may be taken to be 0):
  $E = \{y \in \mathbb{R}^2 : y = tx(1 + u(x)), x \in \mathbb{S}^1, t \in [0,1]\}$,
  $u : \mathbb{S}^1 \to \mathbb{R}$ positive, $\|u\|_{L^\infty} \leq \frac{3}{40}$ and $\|\nabla u\|_{L^\infty} \leq \frac{1}{2}$
  (he also studied higher dimensions!)
- Fuglede (1993): convex sets $\Omega \subset \mathbb{R}^n$

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Theorem (C. Bianchini, G.C., A. Henrot)

There exists a constant $C > 0$ such that $\dfrac{\delta(\Omega)}{\lambda_0(\Omega)^2} \geq C$ for every connected compact set Ω of the plane.

Why connected sets? One can construct $\{\Omega_\varepsilon\}$, non connected, such that $\frac{\delta(\Omega_\varepsilon)}{\lambda_0^2(\Omega_\varepsilon)} \to 0$, as $\varepsilon \to 0$ (two discs of very different radii far from each other).

Why compact sets? We use Blaschke-Lebesgue theorem!

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Why compact sets? We use Blaschke-Lebesgue theorem!

Theorem (C. Gambicchia, A. Pratelli)

For every $D > 0$, there exists a constant $C_N(D) > 0$ such that $\dfrac{\delta(\Omega)}{\lambda_0(\Omega)^2} \geq C_N(D)$ for every set $\Omega \subset \mathbb{R}^N$ with diameter less than $D|E|^{\frac{1}{N}}$.

Thank you for your attention !