The weak boundary between pure and applied mathematics: from perturbation theory to the rotation of the Moon, and back

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–o–

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Tor Vergata

Outline

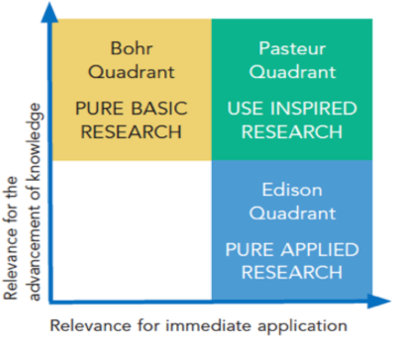
Outline

From theory to applications, and viceversa

• 1944: Franklin Roosvelt asked a report to the Director of the Research and Development Office, Vannevar Bush, on the role that science will have after the war.
Bush: “Basic research is the pacemaker of all technological progress".

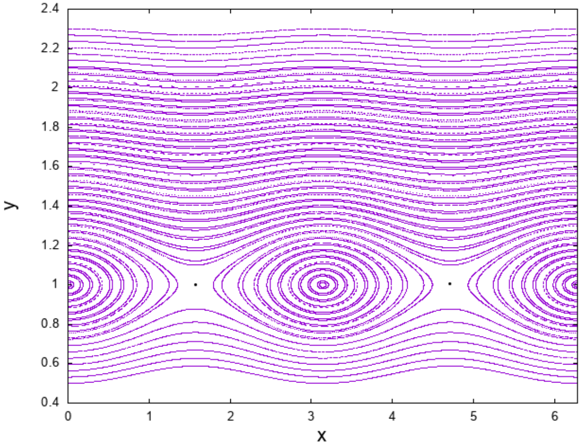
From theory to applications, and viceversa

1997: Pasteur square (Donald E. Stokes, US-NSF advisor).

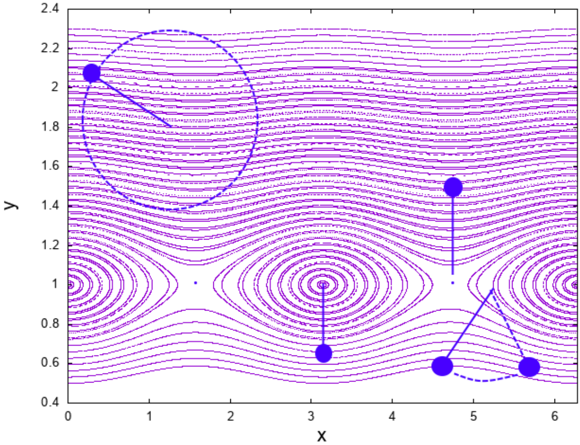


Outline

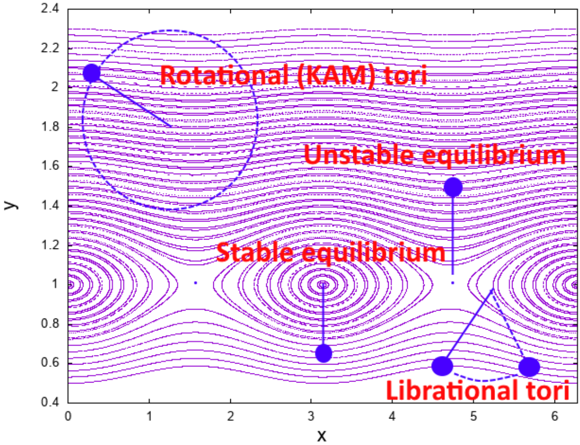
The pendulum



The pendulum, invariant tori and the Moon



The pendulum, invariant tori and the Moon



The pendulum, invariant tori and the Moon

For $(y,x) \in \mathbb{R} \times \mathbb{T}$, $\varepsilon > 0$ small (e.g., $\varepsilon = 0.04$):

$$H(y,x) = \frac{y^2}{2} - \varepsilon\cos(x)$$

or, equivalently, we can consider

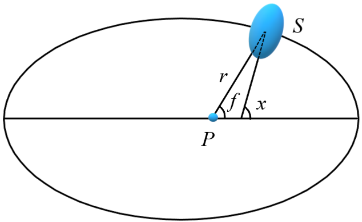
$$H(y,x-t) = \frac{y^2}{2} - \varepsilon\cos(2x-2t)$$

with t being the time.

The pendulum, invariant tori and the Moon

$$H(y,x,t) = \frac{y^2}{2} - \varepsilon\Big[C_2(e)\cos(2x-2t) + C_3(e)\cos(2x-3t) + C_4(e)\cos(2x-4t)\Big]$$

- Conservative spin–orbit problem:
  1. triaxial satellite $\mathcal{S}$ ($I_1 < I_2 < I_3$);
  2. $\mathcal{S}$ moving on a Keplerian orbit around $\mathcal{P}$;
  3. spin–axis perpendicular to the orbit plane;
  4. rigid satellite.



- Here ε represents the equatorial oblateness ($\varepsilon = \frac{3}{2}\frac{I_2-I_1}{I_3}$), e is the orbital eccentricity and the previous picture was for $e = 0$.

The pendulum, invariant tori and the Moon

$$H(y,x,t) = \frac{y^2}{2} - \varepsilon\Big[C_2(e)\cos(2x-2t) + C_3(e)\cos(2x-3t) + C_4(e)\cos(2x-4t)\Big]$$

- Conservative spin–orbit problem:
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$$\begin{aligned}
C_2(e) &= 1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 - \frac{35}{288}e^6 \\
C_3(e) &= \frac{7}{2}e - \frac{123}{16}e^3 + \frac{489}{128}e^5 - \frac{1763}{2048}e^7 \\
C_4(e) &= \frac{17}{2}e^2 - \frac{115}{6}e^4 + \frac{601}{48}e^7 .
\end{aligned}$$

- Here ε represents the equatorial oblateness ($\varepsilon = \frac{3}{2}\frac{I_2-I_1}{I_3}$), e is the orbital eccentricity and the previous picture was for $e = 0$.

The pendulum, invariant tori and the Moon

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• Conservative spin–orbit problem:

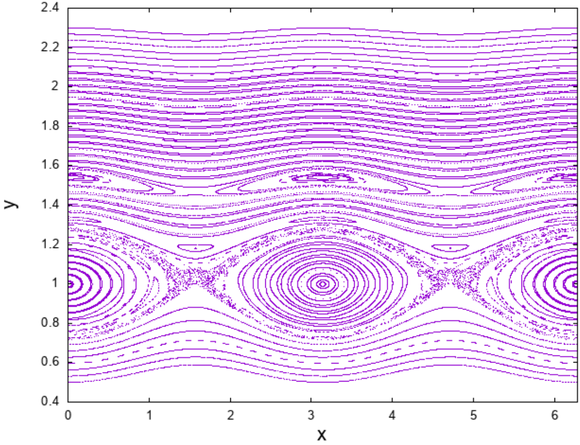
1. triaxial satellite $\mathcal{S}$ ($I_1 < I_2 < I_3$);
2. $\mathcal{S}$ moving on a Keplerian orbit around $\mathcal{P}$;
3. spin–axis perpendicular to the orbit plane;
4. rigid satellite.

• The frequency is $\omega = \frac{\partial H}{\partial y} = y$ and for the Moon $\omega = 1$, which corresponds to the *synchronous or 1:1 spin-orbit resonance* with
period of rotation = period of revolution.

• Here ε represents the equatorial oblateness ($\varepsilon = \frac{3}{2}\frac{I_2 - I_1}{I_3}$), e is the orbital eccentricity and the previous picture was for $e = 0$.

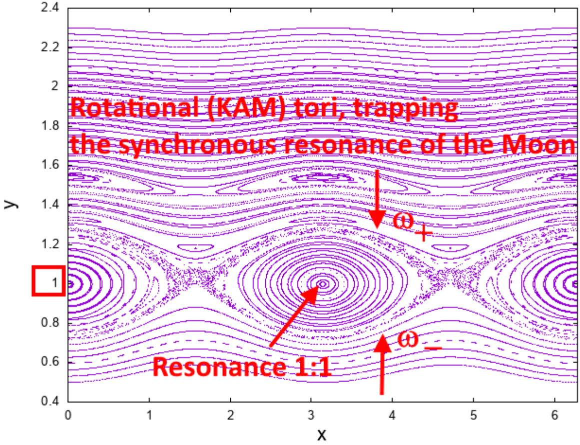
When $e > 0$ (for a fixed ε), we see chaos and less rotational (KAM) tori.

The pendulum, invariant tori and the Moon



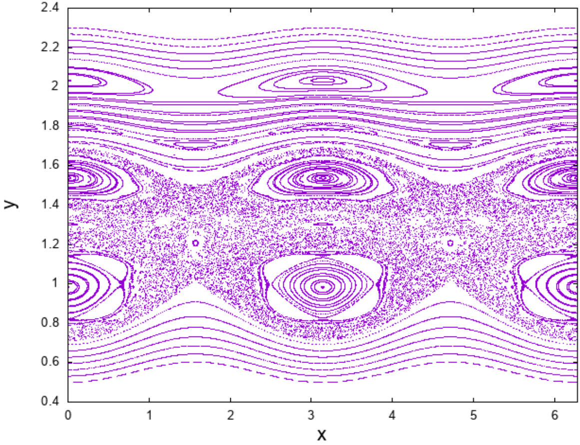
• Eccentricity $e = 0.01$, oblateness parameter $\varepsilon = 0.04$ (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon



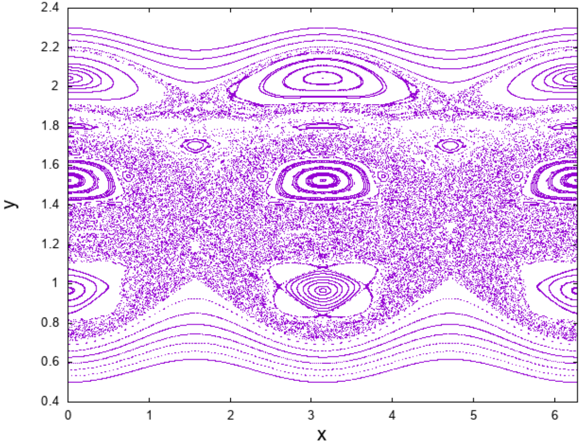
• Eccentricity $e = 0.01$, oblateness parameter $\varepsilon = 0.04$ (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon



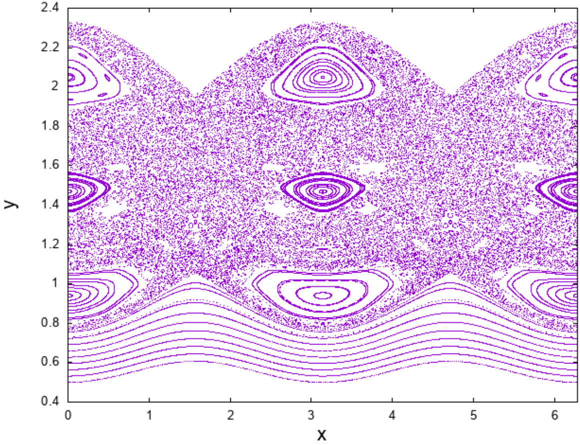
• Eccentricity $e = 0.1$, oblateness parameter $\varepsilon = 0.04$ (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon

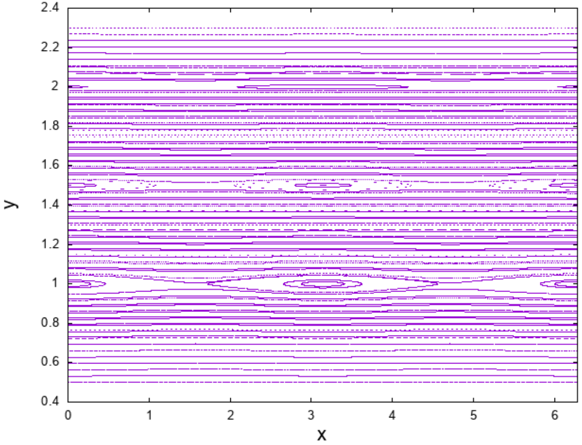


• Eccentricity $e = 0.2$, oblateness parameter $\varepsilon = 0.04$ (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon

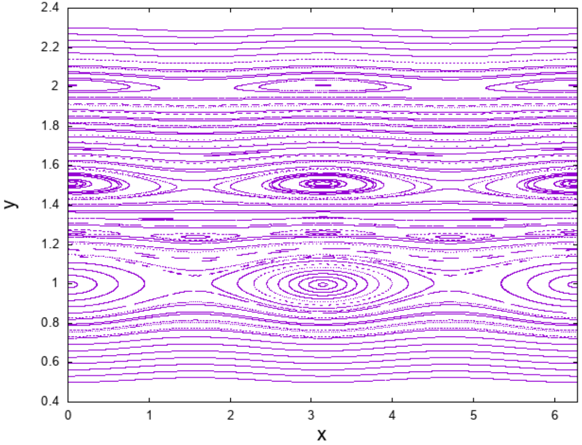


• Eccentricity $e = 0.5$, oblateness parameter $\varepsilon = 0.04$ (Poincaré map for t mod. 2π).



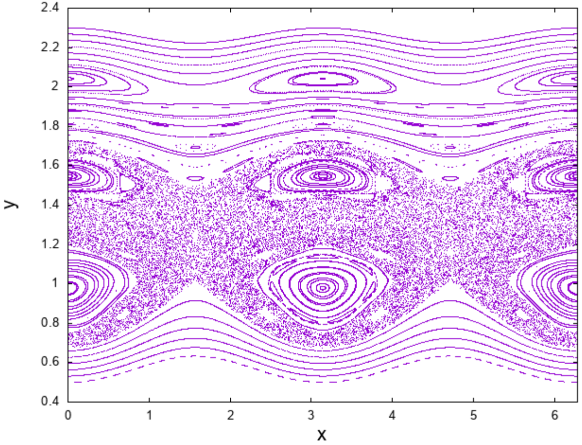
• Oblateness parameter $\varepsilon = 0.001$, eccentricity $e = 0.1$, (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon



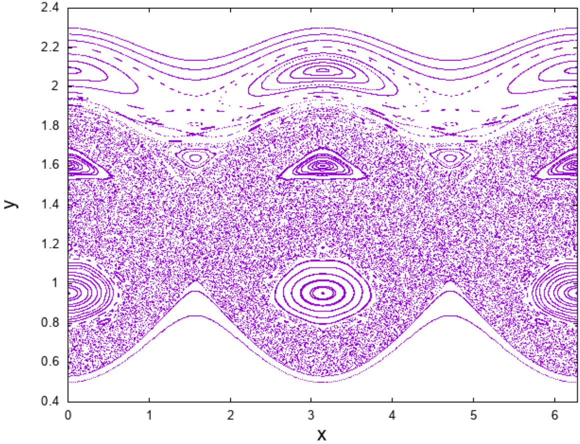
• Oblateness parameter $\varepsilon = 0.01$, eccentricity $e = 0.1$, (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon



• Oblateness parameter $\varepsilon = 0.05$, eccentricity $e = 0.1$, (Poincaré map for t mod. 2π).

The pendulum, invariant tori and the Moon



• Oblateness parameter $\varepsilon = 0.1$, eccentricity $e = 0.1$, (Poincaré map for t mod. 2π).

Can we prove the stability of the rotation of the Moon?

Yes, using Kolmogorov-Arnold-Moser theory.

Outline

KAM theory

- KAM theory: [Kolmogorov1954], [Arnold1963], [Moser1962].

KAM theory: studies the persistence of invariant tori, with fixed frequency, in a nearly-integrable Hamiltonian.

- KAM theory can be developed under two main assumptions:
  - a Diophantine (non resonance) condition on the frequency (to deal with small divisors):

$$|\underline{\omega} \cdot \underline{k}| \geq \frac{1}{C|\underline{k}|^\tau} > 0$$

  - a non–degeneracy condition of the Hamiltonian (to ensure the solution of suitable equations providing a sequence of approximate solutions).

KAM theory: constructive algorithm

• KAM theory: constructive algorithm to give an estimate on ε such that a quasi-periodic torus with fixed frequency ω exists for $\varepsilon \leq \varepsilon_{KAM}(\omega)$.

$\rightsquigarrow$ Does KAM theory provide realistic estimates with $\varepsilon_{KAM}(\omega) \simeq \varepsilon_{astr}(\omega)$ or $\varepsilon_{KAM}(\omega) \simeq \varepsilon_{exp}(\omega)$?

• For a long time, since Hénon application to the 3BP:

$$\varepsilon_{KAM}(\omega) << \varepsilon_{astr}(\omega) < \varepsilon_{exp}(\omega) \ .$$

KAM theory: constructive algorithm

• KAM theory: constructive algorithm to give an estimate on ε such that a quasi-periodic torus with fixed frequency ω exists for $\varepsilon \leq \varepsilon_{KAM}(\omega)$.

$\rightsquigarrow$ Does KAM theory provide realistic estimates with $\varepsilon_{KAM}(\omega) \simeq \varepsilon_{astr}(\omega)$ or $\varepsilon_{KAM}(\omega) \simeq \varepsilon_{exp}(\omega)$? $\Rightarrow$ **YES!**

• For a long time, since Hénon application to the 3BP:

$$\varepsilon_{KAM}(\omega) << \varepsilon_{astr}(\omega) < \varepsilon_{exp}(\omega) \ .$$

Effective KAM theory

▷ a-posteriori approach+automatic reducibility in [Llave, LGJV]): effective KAM estimates, no need to be nearly-integrable and no need to use action-angle variables!

- Main ingredients of the proof:
  - Diophantine condition and non-degeneracy
  - complex extension to get Cauchy estimates; given an analytic function on a domain, Cauchy estimates provide a bound on the norm of the partial derivatives over a smaller domain
  - a Newton quadratic iteration method to find approximate solutions:

$$
\begin{aligned}
&H_0 = Z_0 + \varepsilon R_0 \to \\
&\qquad \to H_1 = Z_1 + \varepsilon^2 R_1 \to \\
&\qquad\qquad \to H_2 = Z_2 + \varepsilon^4 R_2 \to \\
&\qquad\qquad\qquad \to H_3 = Z_3 + \varepsilon^8 R_3
\end{aligned}
$$

...

  - Computer-assisted proofs to get rigorous estimates.

Computer-assisted technique

$\triangleright$ Effective KAM proofs with explicit estimates on parameters imply very long computations $\rightsquigarrow$

$\diamond$ use a computer to perform expansions and computations
$\diamond$ need to control rounding-off and propagation errors

$\Rightarrow$ computer-assisted proof (CAP)

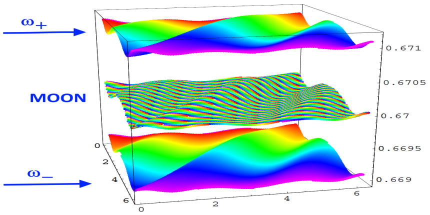
$\triangleright$ CAP: rigorous control of computer errors through, e.g., interval arithmetic (replace real number by intervals and implement an algebra over the intervals).

Spin–orbit: conservative KAM theory

Theorem [A.C. (1990)]

- Consider the conservative spin–orbit Hamiltonian with trigonometric potential R (finite number of Fourier harmonics).
- Fix two frequencies $\omega_- < 1 < \omega_+$, satisfying the Diophantine condition.
- Then, for the true eccentricity of the Moon $e = 0.0549$, there exist invariant tori with frequencies ω_-, ω_+, bounding the motion of the Moon, for any $\varepsilon \leq \varepsilon_{Moon} = 3.45 \cdot 10^{-4}$ (astronomical value).

$\rightsquigarrow$ Hamiltonian in a 3-dim phase space $\Rightarrow$ trapping between 2-dim KAM tori $\Rightarrow$ infinite time stability.

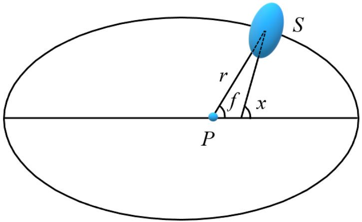


Outline

The dissipative spin-orbit problem

WARNING: satellites are not rigid!

- Dissipative spin–orbit problem:
  1. triaxial satellite $\mathcal{S}$ ($I_1 < I_2 < I_3$);
  2. $\mathcal{S}$ moving on a Keplerian orbit around $\mathcal{P}$;
  3. spin–axis perpendicular to the orbit plane;
  4. tidal torque due to non–rigidity of the satellite.



Dissipative spin–orbit problem

• Conservative spin-orbit equation: 1–dim, time–dependent Hamiltonian:

$$H(y,x,t) = \frac{y^2}{2} + \varepsilon\, R(x,t)\ .$$

• Hamilton's equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\varepsilon R_x(x,t) \end{cases}$$

• Equation of motion:

$$\ddot{x} + \varepsilon R_x(x,t) = 0$$

• Dissipative spin-orbit equation with a dissipation, which is linear in the velocity (and conformally symplectic):

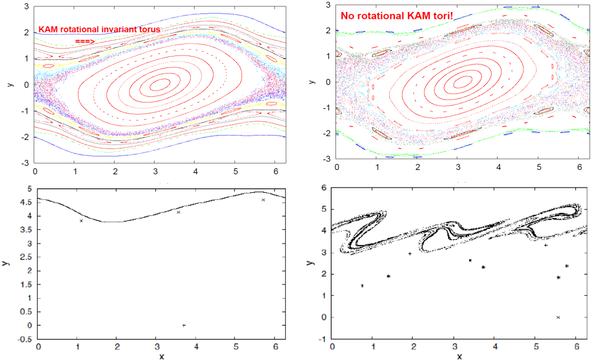
$$\ddot{x} + \varepsilon R_x(x,t) = -\lambda\,(\dot{x} - \mu)$$

Dissipative spin–orbit problem

• KAM theory for dissipative systems:

▷ [Moser1967], see also [Broer, Simó, etc.]

▷ Calleja-Celletti-Llave (JDE 2013): efficient KAM theory for conformally symplectic (dissipative) systems.



• Adding a dissipation to a Hamiltonian system is a very singular perturbation:

- Hamiltonian systems: many quasi-periodic solutions,
- dissipative systems: few attractors + need to include drift parameters.

Outline

Conformally Symplectic KAM theory

Definition

Let $\mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{T}^n$ be a symplectic manifold with symplectic form Ω. A diffeomorphism $f_{\underline{\mu}} : \mathcal{M} \to \mathcal{M}$ is *conformally symplectic*, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that

$$f_{\underline{\mu}}^* \Omega = \lambda \Omega \, .$$

- $\lambda = 1$: symplectic; otherwise we assume λ constant.

Definition

Let a family $f_{\underline{\mu}} : \mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{T}^n \to \mathcal{M}$ of conformally symplectic maps. A ***KAM torus*** with $\underline{\omega} \in \mathcal{D}(C, \tau)$ is a n–dimensional invariant torus described parametrically by an embedding $\underline{K} : \mathbb{T}^n \to \mathcal{M}$ and a drift $\underline{\mu}$ which solve the invariance equation:

$$f_{\underline{\mu}} \circ \underline{K}(\underline{\theta}) = \underline{K}(\underline{\theta} + \underline{\omega}) \, .$$

KAM Theorem (references: Llave-Gonzalez-Jorba-Villanueva & Calleja-Celletti-Llave)

Theorem (CCL, JDE, 2013 (analytic case))

- *Let* $f_{\underline{\mu}}$ *be conformally symplectic,* $\underline{\omega} \in \mathcal{D}(C, \tau)$, $\rho > 0$.
- $(\underline{K}_0, \underline{\mu}_0)$ *approximate solution:*

$$f_{\underline{\mu}_0} \circ \underline{K}_0(\underline{\theta}) = \underline{K}_0(\underline{\theta} + \underline{\omega}) + \underline{E}_0(\underline{\theta}) \ .$$

- *Assume that the solution is sufficiently approximate, i.e.* $\|\underline{E}_0\|_\rho$ *small.*
- *Assume a suitable non–degeneracy condition (on coordinates and parameters).*
- *Then, there exists an exact solution* $(\underline{K}_*, \underline{\mu}_*)$, *such that*

$$f_{\underline{\mu}_*} \circ \underline{K}_*(\underline{\theta}) = \underline{K}_*(\underline{\theta} + \underline{\omega})$$

and for $0 < \delta < \frac{\rho}{2}$*:*

$$\|\underline{K}_* - \underline{K}_0\|_{\rho - 2\delta} \le C_1 \, C^2 \, \delta^{-2\tau} \, \|\underline{E}_0\|_\rho \ , \quad |\underline{\mu}_* - \underline{\mu}_0| \le C_2 \, \|\underline{E}_0\|_\rho \quad (C_1, C_2 > 0) \ .$$

KAM theory for the dissipative spin–orbit problem

Theorem [Calleja-A.C.-Gimeno-Llave, 2022-2024]

- Consider the dissipative spin-orbit problem and consider its Poincaré map.
- Fix a Diophantine frequency, e.g. equal to the golden ratio $\omega = \frac{\sqrt{5}+1}{2}$, $\omega \in \mathcal{D}(C, \tau)$.
- Fix the dissipative factor $\lambda = 10^{-3}$.
- Then, for given parameter values ~~small enough~~, i.e. $\varepsilon = 0.0116$, there exists an invariant attractor with frequency ω.

• Conclusions:

▷ Astronomical value: $\varepsilon_{Moon} = 3.45 \cdot 10^{-4}$.

▷ KAM theoretical value: $\varepsilon_{KAM} = 1.16 \cdot 10^{-2}$.

Outline

Classification of regular and chaotic dynamics

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Classification of regular and chaotic motions in Hamiltonian systems with deep learning

Alessandra Celletti, Catalin Gales, Victor Rodriguez-Fernandez & Massimiliano Vasile

Scientific Reports **12**, Article number: 1890 (2022) | Cite this article

• Classify types of motion of the conservative spin-orbit problem through CNN, starting from time series.

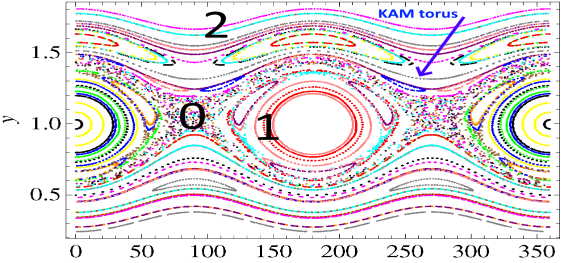
• Use chaos indicators to distinguish between chaotic, rotational, librational motions.

• InceptionTime CNN: at each of the 5 layers of a CNN, it applies a set of convolution operations to the time series, which creates a transformation of the input data, so that the series becomes easier to be classified.

Classification of regular and chaotic dynamics

• Chaos indicators: Fast Lyapunov Indicators (FLI) and Frequency Map Analysis (FMA).

1. Given an initial condition, generate a finite number of time series as the solution at different time intervals with fixed step size.
2. For each time series, compute FLI or FMA and assign a label: chaotic $\mapsto$ 0, librational $\mapsto$ 1, rotational $\mapsto$ 2.
3. InceptionTime: train on some data sets, validate and complete the cartography (0-1-2).



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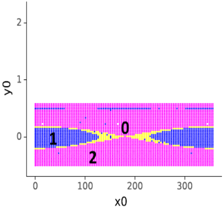
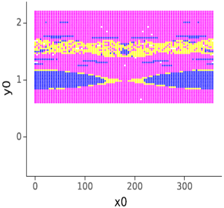
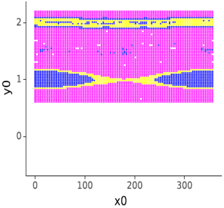


(a) Map of the training set



(b) Map of the test set coloured with the true labels



(c) Map of the test set coloured with the DL predictions

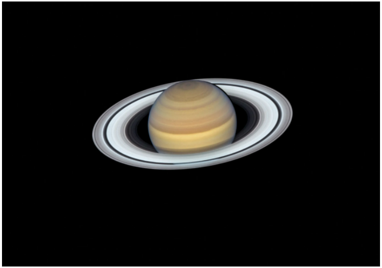
Outline

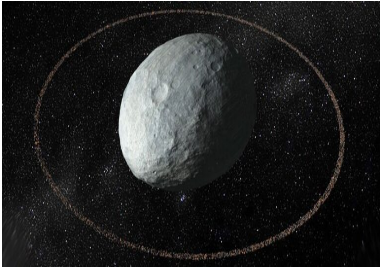
What next?

• Among possible other subjects, we can consider the dynamics of a ring (massless) particle around a central body:

▷ planets: well approximated by ellipsoids with small oblateness and elongation (planets with rings: Saturn, Jupiter, Uranus, Neptune)

▷ small bodies: have irregular shape (small bodies with rings: dwarf planets (136108) Haumea, (50000) Quaoar and the Centaur (10199) Chariklo).





Rings

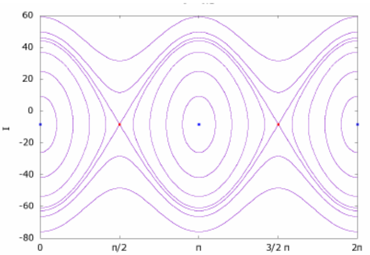
• We can model the small body as:

▷ triaxial ellipsoid with high oblateness $(I_2 - I_1)/I_3$ and elongation $(2I_3 - I_2 - I_1)/I_3$

▷ topographic feature (sphere with a mass anomaly on the equator)

• Haumea, Chariklo, Quaoar have rings in a 1:3 spin-orbit resonance! (rotational/orbital periods=1/3)

Why not 1:1, 1:2, 2:5, 2:3, etc.?



Rings

• To answer we need:

▷ good model (gravitational potential expanded in spherical harmonics)
▷ good variables (epicyclic variables) allowing to properly define corotation and Lindblad resonances
▷ perturbation theory to expand around the resonance
▷ stability analysis of the equilibria
▷ bifurcation theory.

Ongoing works with Sara Di Ruzza and Irene De Blasi.

Word count


orbit
satellite
parameter
eccentricity
Hamiltonian
invariant
KAM
Moon
theory
can
fixed
exists
non
Poincarè
spin
tori
time
frequency
map
rotational
oblateness
consider
small
Oblateness
around
conformally
pendulum
condition
dissipative
symplectic

Main references

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• Calleja R., Celletti A., Gimeno J., de la Llave R., "A map reduction and KAM tori construction for the dissipative spin-orbit problem", J. Nonlinear Science (2022)

• Calleja R., Celletti A., Gimeno J., de la Llave R., "KAM estimates in the dissipative spin-orbit problem", CNSNS (2022)

• Calleja R., Celletti A., Gimeno J., de la Llave R., "Breakdown threshold of invariant attractors in the dissipative spin-orbit problem", J. Nonlinear Science (2024)

▷ Machine learning:

• Celletti A., Gales C., Rodriguez V. Vasile M., "Classification of Regular and Chaotic Motions in Hamiltonian systems with Deep Learning", Scientific Reports (2022)