Double phase problems with different boundary conditions

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A differential operator that has found a place in many research fields in recent years is the so-called "double phase operator", which is defined by

$$u \mapsto -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right), \tag{1}$$

for any function u belonging to a suitable space and $1 < p < q < \infty$.

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for any function u belonging to a suitable space and $1 < p < q < \infty$. The associated energy functional is given by

$$I(u) = \int_\Omega H(x, \nabla u)\mathrm{d}x = \int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x, \tag{2}$$

and its integrand has unbalanced growth if $\mu \in L^\infty(\Omega), \mu \geq 0$, namely

$$b_1|\xi|^p \leq H(x,\xi) \leq b_2|\xi|^q \quad \text{for a. a. } x \in \Omega, \forall\, \xi \in \mathbb{R}^N, b_1, b_2 > 0. \tag{3}$$

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A classical example of balanced growth is the p-Laplacian operator: $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}(a(x, \nabla u))$

$$(a(x,\xi)) \cdot \xi = \xi^p$$

Zhikov was the first who studied this functional to describe the behaviour of strongly anisotropic materials.

$$-\operatorname{div}\left(\underbrace{|\nabla u|^{p-2}\nabla u}_{\text{material 1}}+\underbrace{\mu(x)}_{\text{geometry}}\underbrace{|\nabla u|^{q(x)-2}\nabla u}_{\text{material 2}}\right),$$

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$$-\operatorname{div}\left(\underbrace{|\nabla u|^{p-2}\nabla u}_{\text{material 1}} + \underbrace{\mu(x)}_{\text{geometry}} \underbrace{|\nabla u|^{q(x)-2}\nabla u}_{\text{material 2}}\right),$$

Moreover, the double phase operator arises also in the context of the Lavrentiev gap phenomenon, the thermistor problem and the duality theory.

M. Colombo, G. Mingione,
Regularity for double phase variational problems
Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.

V.V. Zhikov,
Averaging of functionals of the calculus of variations and elasticity theory
Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.

V.V. Zhikov,
On variational problems and nonlinear elliptic equations with nonstandard growth conditions
J. Math. Sci. **173** (2011), no. 5, 463–570.

Furthermore, one can consider the double phase operator with variable exponents $p(\cdot)$ and $q(\cdot)$, i.e.

$$u \mapsto -\operatorname{div}\underbrace{\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)}_{\mathcal{F}(u)}, \tag{4}$$

which is still related to applied problems, as for instance the transonic flow.

A. Bahrouni, V.D. Rădulescu, D.D. Repovš,
Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves
Nonlinearity **32** (2019), no. 7, 2481–2495.

M.A. Ragusa, A. Tachikawa,
Regularity for minimizers for functionals of double phase with variable exponents
Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.

In recent years, many authors have shown existence and multiplicity results for double phase problems with constant exponents p and q.

S. Biagi, F. Esposito, E. Vecchi,
Symmetry and monotonicity of singular solutions of double phase problems
J. Differential Equations **280** (2021), 435–463.

L. Gasiński, N.S. Papageorgiou,
Constant sign and nodal solutions for superlinear double phase problems
Adv. Calc. Var. **14** (2021), no. 4, 613–626.

L. Gasiński, P. Winkert,
Existence and uniqueness results for double phase problems with convection term
J. Differential Equations **268** (2020), no. 8, 4183–4193.

N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš,
Double-phase problems and a discontinuity property of the spectrum
Proc. Amer. Math. Soc. **147** (2019), no. 7, 2899–2910.

K. Perera, M. Squassina,
Existence results for double-phase problems via Morse theory
Commun. Contemp. Math. **20** (2018), no. 2, 1750023, 14 pp.

Whereas, in the variable exponent case the results are fewer than the constant case.

J. Cen, S.J. Kim, Y.-H. Kim, S. Zeng,
Multiplicity results of solutions to the double phase anisotropic variational problems involving variable exponent
Adv. Differential Equations **28** (2023), 5-6, 467–504.

S. Leonardi, N.S. Papageorgiou,
Anisotropic Dirichlet double phase problems with competing nonlinearities
Rev. Mat. Complut. **36** (2023), 469–490.

J. Liu, P. Pucci,
Existence of solutions for a double-phase variable exponent equation without the Ambrosetti-Rabinowitz condition
Adv. Nonlinear Anal. **12** (2023), no. 1, anona–2022–0292.

S. Zeng, V.D. Rădulescu, P. Winkert,
Double phase obstacle problems with variable exponent
Adv. Differential Equations **27** *(2022), no. 9-10, 611–645.*

Preliminaries

Calculus of variations and critical point theory

Problem $\rightsquigarrow$ Functional

Calculus of variations and critical point theory

Problem $\rightsquigarrow$ Functional

Solutions $\iff$ Critical points

Calculus of variations and critical point theory

Problem $\rightsquigarrow$ Functional

Solutions $\Longleftrightarrow$ Critical points

In classical theory, this means considering a functional J defined on a Banach space that admits derivative according to Gâteaux $J' : X \to X^*$ and solving the Euler equation

$$J'(x) = 0.$$

Calculus of variations and critical point theory

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$$J'(x) = 0.$$

How can we find the critical points?

- Global minimum $\rightsquigarrow$ Direct methods theorem;
- Saddle point $\rightsquigarrow$ Mountain Pass theorem.

A. Ambrosetti, P.H. Rabinowitz
Dual variational methods in critical point theory and applications
Journal of Functional Analysis **14** *4 (1973), pp. 349–381*

- Local minima $\rightsquigarrow$ Ricceri, Bonanno, Candito, D'Aguì....

The variational formulation that is considered is

$$I = \Phi - \Psi$$

where Φ and Ψ are continuously Gâteaux differentiable functionals defined on an infinite dimensional real Banach space.

G. Bonanno, P.Candito
Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities
J. Differential Equations **244** 12, (2008), pp. 3031–3059.

G. Bonanno
A critical point theorem via the Ekeland variational principle
Nonlinear Anal. **75**, (2012), pp. 2992–3007.

G. Bonanno, G. D'Aguì
Two non-zero solutions for elliptic Dirichlet problems
Z. Anal. Anwend. **35** (2016), no. 4, pp. 449–464.

For any $r \in C(\overline{\Omega})$, we set

$$r_+ := \max_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_- := \min_{x\in\overline{\Omega}} r(x), \tag{5}$$

and define

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}) \,:\, r_- > 1\}.$$

Hypotheses (H)

- $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ is a bounded domain with Lipschitz boundary $\partial\Omega$;
- $p, q \in C(\overline{\Omega})$ such that

$$1 < p(x) < N \quad \text{and} \quad p(x) < q(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

  where $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$ is the critical Sobolev exponent to $p(\cdot)$;
- $\mu \in L^\infty(\Omega)$, with $\mu(\cdot) \geq 0$.

Let $\mathcal{H}\colon \Omega \times [0,\infty[\to [0,\infty[$ be the nonlinear function defined by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \quad \text{for all } (x,t) \in \Omega \times [0,\infty[, \tag{6}$$

and let $\rho_{\mathcal{H}}(\cdot)$ be the corresponding modular defined by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\,\mathrm{d}x, \tag{7}$$

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Then, we denote by $L^{\mathcal{H}}(\Omega)$ the Musielak-Orlicz space, given by

$$L^{\mathcal{H}}(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable} : \rho_{\mathcal{H}}(u) < +\infty\}, \tag{8}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1\right\}. \tag{9}$$

We denote by $W^{1,\mathcal{H}}(\Omega)$ the Musielak-Orlicz Sobolev space defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \, : \, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}, \tag{10}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}, \tag{11}$$

with $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$ and by $W_0^{1,\mathcal{H}}(\Omega)$ we indicate the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$.

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$L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are separable and reflexive Banach spaces.

Moreover, $W_0^{1,\mathcal{H}}(\Omega)$ satisfies a Poincaré inequality and

$$\|u\|_{1,\mathcal{H},0} := \|\nabla u\|_{\mathcal{H}}$$

is an equivalent norm

Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert,
A new class of double phase variable exponent problems: Existence and uniqueness
J. Differential Equations **323** (2022), 182–228.

We point out that the exponents $p(\cdot)$ and $q(\cdot)$ do not need to verify a condition of the type

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{1}{N} \tag{12}$$

as it was needed, for example, in Kim-Kim-Oh-Zeng or in Colasuonno-Squassina and Liu-Dai for the constant exponent case. Indeed, it's only required that

$$p(\cdot) < q(\cdot) < p^*(\cdot), \tag{13}$$

F. Colasuonno, M. Squassina,
Eigenvalues for double phase variational integrals
Ann. Mat. Pura Appl. (4) **195** (2016), no. 6, 1917–1959.

I.H. Kim, Y.-H. Kim, M.W. Oh, S. Zeng,
Existence and multiplicity of solutions to concave-convex-type double-phase problems with variable exponent
Nonlinear Anal. Real World Appl. **67** (2022), Paper No. 103627, 25 pp.

W. Liu, G. Dai,
Existence and multiplicity results for double phase problem
J. Differential Equations **265** (2018), no. 9, 4311–4334.

Dirichlet problem

Consider the following double phase problem with variable exponents

$$\begin{aligned} -\operatorname{div}\mathcal{F}(u) &= \lambda f(x,u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \qquad (D_\lambda)$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\lambda > 0$ is a parameter and we assume and p, q, μ satisfy assumption (H).

In this paper, we establish the existence of two bounded weak solutions with opposite energy sign through a critical points theorem due to Bonanno-D'Aguì-2016.

E. Amoroso, G. Bonanno, G. D'Aguí, P. Winkert

Two solutions for Dirichlet double phase problems with variable exponents

Advanced Nonlinear Studies (2024), doi: 10.1515/ans-2023-0134

Assumptions on the perturbation (H_f^D)

Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ be such that:

(f_1^D) f is a Carathéodory function;

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(f_2^D) there exist $\ell \in C_+(\overline{\Omega})$ with $\ell_+ < (p_-)^*$ and $\kappa_1 > 0$ such that

$$|f(x,t)| \le \kappa_1 \left(1 + |t|^{\ell(x)-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R};$$

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(f_3^D) $\displaystyle\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^{q_+}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$

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(f_3^D) $\displaystyle\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^{q_+}} = +\infty$ uniformly for a.a. $x \in \Omega$;

(f_4^D) there exists $\zeta \in C_+(\overline{\Omega})$, with $\zeta_- \in \left((\ell_+ - p_-)\frac{N}{p_-}, \ell_+\right)$, and $\zeta_0 > 0$ such that

$$0 < \zeta_0 \leq \liminf_{t\to\pm\infty} \frac{f(x,t)t - q_+ F(x,t)}{|t|^{\zeta(x)}} \quad \text{uniformly for a.a. } x \in \Omega.$$

Variational setting

Space: $X = W_0^{1,\mathcal{H}}(\Omega)$, $\|u\|_X = \|\nabla u\|_{\mathcal{H}}$.

Functionals: Consider $\Phi, \Psi, I_\lambda : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) := \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \right) \mathrm{d}x$$

$$\Psi(u) := \int_\Omega F(x,u) \mathrm{d}x$$

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u).$$

$u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of (D_λ)

$$\Updownarrow$$

u is a critical point of I_λ, i.e. $\langle I_\lambda'(u), v \rangle = 0$ for any $v \in W_0^{1,\mathcal{H}}(\Omega)$.

Theorem (A.-Bonanno-D'Aguì-Winkert)

Assume that (H) and (H_f^D) hold.
Furthermore, suppose that f is nonnegative and

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty \qquad (h_3^D)$$

Then, there exists $\lambda^* > 0$ such that for each $\lambda \in]0, \lambda^*[$ problem (D_λ) admits at least two nontrivial and nonnegative bounded weak solutions $u_{\lambda,1}$, $u_{\lambda,2} \in W_0^{1,\mathcal{H}}(\Omega)$ with opposite energy sign.

Main Tool: Theorem Bonanno-D'Aguì (2016)

Let X be a real Banach space and let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.
Assume that Φ is coercive and there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{14}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (C)-condition and it is unbounded from below.

Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

G. Bonanno and G. D'Aguì,
Two non-zero solutions for elliptic Dirichlet problems,
Zeitschrift für Analysis und ihre Anwendung, **35** (2016), 449-464.

Sketch of the proof - Existence

- from $\|u\|_{1,\mathcal{H},0} \to +\infty \iff \rho_{1,\mathcal{H},0}(u) \to +\infty$, it follows that Φ is coercive;

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- I_λ satisfies the (C)-condition for all $\lambda > 0$ from $(f_2^D), (f_4^D)$.

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- I_λ satisfies the (C)-condition for all $\lambda > 0$ from $(f_2^D), (f_4^D)$.
- From (h_3^D) follows that there exist $r \in \mathbb{R}$ and $\tilde{u} \in W_0^{1,\mathcal{H}}(\Omega)$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{15}$$

where

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \frac{2\eta}{R}(R - |x - x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \eta & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases} \tag{16}$$

Sketch of the proof - Boundedness

From the paper of Crespo-Blanco-Winkert it follows that $u_{\lambda,1}$, $u_{\lambda,2}$ belong to $L^\infty(\Omega)$. Indeed, they proved that any weak solution u of the following generic problem

$$\begin{aligned} -\operatorname{div} \mathcal{A}(x, u, \nabla u) &= \mathcal{B}(x, u, \nabla u) \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{17}$$

is such that $u \in L^\infty(\Omega)$, under more general growth assumption.

Á. Crespo-Blanco, P. Winkert,
Nehari manifold approach for superlinear double phase problems with variable exponents
Annali di Matematica, **203**, 605–634 (2024).

Example

Consider $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,t) = \begin{cases} |t|^{\alpha(x)-2}t & \text{if } |t| < 1, \\ |t|^{\beta(x)-2}t\,(\log|t|+1) & \text{if } |t| \geq 1, \end{cases}$$

where $\alpha, \beta \in C(\overline{\Omega})$ such that $q_+ < \beta(x) < (p_-)^*$ for all $x \in \overline{\Omega}$ and

$$\frac{\beta_+}{p_-} - \frac{\beta_-}{N} < 1.$$

Then, f satisfies assumptions (H_f^D) and we can apply the previous theorem with $\tilde{f}(x,t) = |f(x,t)|$ for every $(x,t) \in \Omega \times \mathbb{R}$, requiring also that $\alpha(x) < p_-$ for all $x \in \overline{\Omega}$.

Neumann problem

E. Amoroso, Á. Crespo-Blanco, P. Pucci, P. Winkert
Superlinear elliptic equations with unbalanced growth and nonlinear boundary condition
Bulletin des Sciences Mathématiques, **197** (2024), doi: 10.1016/j.bulsci.2024.103534.

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In this paper we prove:

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In this paper we prove:

- the existence of a new equivalent norm in Musielak-Orlicz Sobolev spaces in a very general setting;
- a new result on the boundedness of the solutions for a wide class of nonlinear Neumann problems;
- the existence of multiple solutions for a variable exponent double phase problem with a nonlinear boundary condition under very general assumptions on the nonlinearities.

A new equivalent norm

Set

$$\|u\|_{1,\mathcal{H}}^{*} = \inf\Bigg\{\tau > 0 \,:\, \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)}\right) \mathrm{d}x + \int_\Omega \vartheta_1(x)\left|\frac{u}{\tau}\right|^{\delta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \vartheta_2(x)\left|\frac{u}{\tau}\right|^{\delta_2(x)} \mathrm{d}\sigma \le 1\Bigg\},$$

where, in addition to (H), we suppose the following conditions

Hypotheses (H_1)

1. $\delta_1, \delta_2 \in C(\overline{\Omega})$ with $1 \le \delta_1(x) \le p^*(x)$ and $1 \le \delta_2(x) \le p_*(x)$ for all $x \in \overline{\Omega}$, where

   (a$_1$) $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if $\delta_1(x) = p^*(x)$ for some $x \in \overline{\Omega}$;
   (a$_2$) $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, if $\delta_2(x) = p_*(x)$ for some $x \in \overline{\Omega}$;
2. $\vartheta_1 \in L^\infty(\Omega)$ with $\vartheta_1(x) \ge 0$ for a.a. $x \in \Omega$;
3. $\vartheta_2 \in L^\infty(\partial\Omega)$ with $\vartheta_2(x) \ge 0$ for a.a. $x \in \partial\Omega$;
4. $\vartheta_1 \not\equiv 0$ or $\vartheta_2 \not\equiv 0$.

Finally, denote by $B\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ the nonlinear operator defined pointwise by

$$\begin{aligned}\langle B(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x \\ &+ \int_\Omega \vartheta_1(x)|u|^{\delta_1(x)-2}uv\,\mathrm{d}x + \int_{\partial\Omega} \vartheta_2(x)|u|^{\delta_2(x)-2}uv\,\mathrm{d}\sigma,\end{aligned} \tag{18}$$

for all $u, v \in W^{1,\mathcal{H}}(\Omega)$.

Proposition 2 (E.A.-Á. Crespo-Blanco-P. Pucci-P. Winkert)

Let hypotheses (H) and $(H1)$ be satisfied. Then, the operator $B\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ is bounded, continuous, strictly monotone and of type (S_+), that is,

$$\text{if} \quad u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle B(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$ in $W^{1,\mathcal{H}}(\Omega)$.

Neumann problem - Three solutions

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$ and denoting with $\nu(x)$ the outer unit normal of Ω at $x \in \partial\Omega$, we study the following problem

$$\begin{aligned} -\operatorname{div} \mathcal{F}(u) + |u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega, \\ \mathcal{F}(u) \cdot \nu &= g(x,u) - |u|^{p(x)-2}u && \text{on } \partial\Omega, \end{aligned} \tag{N}$$

where

Hypotheses (H^N)

$p, q \in C(\overline{\Omega})$ such that

$$1 < p(x) < N \quad \text{and} \quad p(x) < q(x) \leq q_+ < (p_-)_* \quad \text{for all } x \in \overline{\Omega},$$

with $p(\cdot)_* = \frac{(N-1)p(\cdot)}{N-p(\cdot)}$ and $\mu \in L^\infty(\Omega)$, with $\mu(\cdot) \geq 0$.

Robin problem

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$ and denoting with $\nu(x)$ the outer unit normal of Ω at $x \in \partial\Omega$, we study the following problem

$$\begin{aligned} -\operatorname{div}\mathcal{F}(u) + \alpha(x)|u|^{p(x)-2}u &= \lambda f(x,u) && \text{in } \Omega, \\ \mathcal{F}(u)\cdot\nu &= -\beta(x)|u|^{p_*(x)-2}u && \text{on } \partial\Omega, \end{aligned} \qquad (R_\lambda)$$

where $\lambda > 0$ and

Assumption (H^R)

$p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, $q \in C(\overline{\Omega})$ such that

$$1 < p(x) < N, \quad p(x) < q(x) < p_*(x) \quad \text{for all } x \in \overline{\Omega},$$

$\mu \in L^\infty(\Omega)$ with $\mu \geq 0$ a.e. in Ω, $\alpha \in L^\infty(\Omega)$ with $\alpha \geq 0$ a.e. in Ω and $\alpha \not\equiv 0$, $\beta \in L^\infty(\partial\Omega)$ with $\beta \geq 0$ a.e. in $\partial\Omega$.

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We endow the space $W^{1,\mathcal{H}}(\Omega)$ with the following equivalent norm

$$\|u\| = \inf\left\{\tau > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)}\right)\mathrm{d}x + \int_\Omega \alpha(x)\left|\frac{u}{\tau}\right|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} \beta(x)\left|\frac{u}{\tau}\right|^{p_*(x)} \mathrm{d}\sigma \ \leq 1\right\},$$

which is obtained by $\|\cdot\|^*_{1,\mathcal{H}}$ by choosing $\vartheta_1 \equiv \alpha, \delta_1 \equiv p, \vartheta_2 \equiv \beta$ and $\delta_2 \equiv p_*$.

$$\|u\|^*_{1,\mathcal{H}} = \inf\left\{\tau > 0 \ : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\tau}\right|^{q(x)}\right) \mathrm{d}x + \int_\Omega \vartheta_1(x)\left|\frac{u}{\tau}\right|^{\delta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \vartheta_2(x)\left|\frac{u}{\tau}\right|^{\delta_2(x)} \mathrm{d}\sigma \leq 1\right\},$$

Assumptions on the perturbation (H_f^R)

Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ for all $x \in \Omega$ be such that:

(f_1^R) f is L^1-Carathéodory;

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(f_1^R) f is L^1-Carathéodory;

(f_2^R) there exist $k_1, k_2 > 0$ and $\ell \in C_+(\overline{\Omega})$ with $\ell_+ < (p^*)_-$, such that

$$|f(x,t)| \leq k_1 + k_2 |t|^{\ell(x)-1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

Assumptions on the perturbation (H_f^R)

Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ for all $x \in \Omega$ be such that:

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$$|f(x,t)| \le k_1 + k_2 |t|^{\ell(x)-1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(AR) there exist $\eta > (p_*)_+$, $s > 0$ such that

$$0 < \eta F(x,t) \le t f(x,t)$$

for all $x \in \Omega$ and for all $t \ge s$.

Variational setting

Space: $X = W^{1,\mathcal{H}}(\Omega), \|u\|_X = \|u\|$.

Functionals: Consider $\Phi, \Psi, I_\lambda : W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\Phi(u) :=& \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \right) \mathrm{d}x + \int_\Omega \alpha(x)\frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}x \\
&+ \int_{\partial\Omega} \beta(x)\frac{|u|^{p_*(x)}}{p_*(x)}\,\mathrm{d}\sigma, \\
\Psi(u) :=& \int_\Omega F(x,u)\mathrm{d}x \\
I_\lambda(u) :=& \Phi(u) - \lambda\Psi(u),
\end{aligned}$$

$u \in W^{1,\mathcal{H}}(\Omega)$ is a weak solution of (R_λ)

$\Updownarrow$

u is a critical point of I_λ, i.e. $\langle I'_\lambda(u), v\rangle = 0$ for any $v \in W^{1,\mathcal{H}}(\Omega)$.

Theorem (A.-Morabito)

Let (H^R) , (H_f^R) be satisfied.
Suppose that f is nonnegative and

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p_-}} = +\infty. \tag{H_5}$$

Then, there exists $\lambda^* > 0$ such thtat for each $\lambda \in]0, \lambda^*[$ problem (R_λ) admits at least two nontrivial and nonnegative weak solutions with opposite energy sign.

Example

Consider the following function

$$f(x,t) = h_1(x) + h_2(x)|t|^{\xi(x)-1} \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{19}$$

with $h_1, h_2 \in L^\infty(\Omega)$, $\operatorname{essinf}_\Omega h_i > 0$ for $i = 1, 2$ and $\xi \in C_+(\overline{\Omega})$ with

$$(p_*)_+ < \xi(x) < (p^*)_- \quad \text{for all } x \in \overline{\Omega}.$$

We have that

$$F(x,t) = h_1(x)t + \frac{h_2(x)}{\xi(x)} t^{\xi(x)} \quad \text{for all } x \in \Omega, t > 0,$$

satisfies (AR)-condition for any $(p_*)_+ < \eta \leq \xi_-$ and assumption (H_5) of the previous Theorem.

Thanks for your kind attention.